

Long-wavelength limit of gyrokinetics in a turbulent tokamak and its intrinsic ambipolarity

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Abstract. Recently, the electrostatic gyrokinetic Hamiltonian and change of coordinates have been computed to order ϵ^2 in general magnetic geometry. Here ϵ is the gyrokinetic expansion parameter, the gyroradius over the macroscopic scale length. Starting from these results, the long-wavelength limit of the gyrokinetic Fokker-Planck and quasineutrality equations is taken for tokamak geometry. Employing the set of equations derived in the present article, it is possible to calculate the long-wavelength components of the distribution functions and of the poloidal electric field to order ϵ^2 . These higher-order pieces contain both neoclassical and turbulent contributions, and constitute one of the necessary ingredients (the other is given by the short-wavelength components up to second order) that will eventually enter a complete model for the radial transport of toroidal angular momentum in a tokamak in the low flow ordering. Finally, we provide an explicit and detailed proof that the system consisting of second-order gyrokinetic Fokker-Planck and quasineutrality equations leaves the long-wavelength radial electric field undetermined; that is, the turbulent tokamak is intrinsically ambipolar.

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1. Introduction

Gyrokinetic theory [1] and gyrokinetic codes [2, 3, 4, 5, 6, 7] are recognized as the fundamental tools for the description of microturbulence in fusion and astrophysical plasmas. Gyrokinetic theory consists of the elimination of the degree of freedom associated to the gyration of the charged particle around the magnetic field order by order in an asymptotic expansion in $\epsilon = \rho/L \ll 1$, where ρ is the gyroradius and L is the macroscopic scale length of the problem. This procedure reduces the phase-space dimension and, more importantly, the degree of freedom averaged out is precisely the one with the shortest time scale. The savings in computational time that gyrokinetics has provided have made it possible to simulate kinetic plasma turbulence. Derivations of the gyrokinetic equations by iterative methods can be found in references [8, 9, 10, 11, 12], and via Hamiltonian and Lagrangian methods in references [13, 14, 15, 16]. A recent review of gyrokinetic theory is given in [17].

The gyrokinetic equations have typically been solved only for the turbulent components of the distribution function and the electrostatic potential (we restrict our discussion to electrostatic gyrokinetics), but in recent years growing supercomputer capabilities have motivated an increasing interest in the extension of gyrokinetic calculations to longer wavelengths and transport time scales. However, at least for a tokamak ‡, this is a subtle issue, as F. I. Parra and P. J. Catto have discussed in a series of papers [12, 18, 19, 20, 21, 22]. The main lines of the argument can be stated in a succinct way. The perpendicular component of the long-wavelength piece of the plasma velocity depends on the long-wavelength radial electric field through the $\mathbf{E} \times \mathbf{B}$ drift. The momentum conservation equation can be used to obtain the three components of the velocity, and from it, derive the radial electric field. The plasma velocity is to lowest order parallel to the flux surfaces because the radial particle drift is small. Then, the poloidal and toroidal components of the momentum conservation equation are sufficient to calculate the velocity to the order of interest, and by decomposing it in parallel and perpendicular components, the radial electric field can be obtained by making the perpendicular component equal to the $\mathbf{E} \times \mathbf{B}$ drift plus the diamagnetic velocity. The poloidal component of the velocity is strongly damped by collisions because the poloidal direction is not a direction of symmetry. The poloidal velocity is determined by setting the collisional viscosity in the poloidal direction equal to zero, giving a poloidal velocity proportional to the ion temperature gradient unless collisionality is really small and turbulence can compete with the collisional damping [18, 22]. Unfortunately, the toroidal component of the momentum equation that would give the toroidal component of the velocity and completely determine the radial electric field is identically satisfied to order ϵ^2 by any toroidal velocity [18, 20]. Since gyrokinetic equations are customarily derived and solved to order ϵ , the tokamak long-wavelength radial electric field cannot be correctly obtained from the standard set of gyrokinetic equations available in the literature.

‡ Throughout this paper “tokamak” means “axisymmetric tokamak”.

In the limit in which the velocity is of the order of the diamagnetic velocity, known as low flow limit, the calculation of the radial flux of toroidal angular momentum, which we need to compute the radial electric field, is especially demanding because this flux is smaller than the radial flux of particles and energy in the expansion in ϵ . The low flow limit is relevant in the study of intrinsic rotation [23, 24, 25]. In references [22, 24], a method to calculate the toroidal angular momentum conservation equation in the low flow limit to the order in which it is not identically zero is proposed. With the toroidal angular momentum equation to this order, it is possible to obtain the toroidal rotation and hence calculate the radial electric field. The formula for the radial flux of toroidal angular momentum in [22, 24] is given as a sum of several integrals over the first- and second-order pieces of the distribution functions and the electrostatic potential. To avoid calculating these second-order pieces in complete detail, a subsidiary expansion in $B_p/B \ll 1$ was employed, where B_p is the poloidal magnetic field and B is the total magnetic field. With the derivation for the first time of the gyrokinetic equations and change of coordinates in general magnetic geometry up to second order [16], it has become possible to calculate the second-order pieces without resorting to a subsidiary expansion. In this article, we present the equations that need to be solved to obtain the long-wavelength second order pieces. These equations have not been explicitly written before. They contain neoclassical [26, 27] and turbulent contributions. The turbulent contributions have never been considered to our knowledge, and the complete neoclassical equations have only been used in the Pfirsch-Schlüter limit in [28]. Calculations of the neoclassical radial flux of toroidal angular momentum in other collisionality regimes have relied on the $B_p/B \ll 1$ expansion [29].

We emphasize that the equations derived here are the first step towards a complete model for the computation of radial transport of toroidal angular momentum in a tokamak. The second step, that will be taken in a future publication, includes the derivation of the equations determining the short-wavelength components of the distribution functions and electrostatic potential to second order. To ease the reading of the paper, we advance in this introduction which are the equations that we derive, and that will eventually enter the aforementioned complete model for toroidal angular momentum transport in a tokamak. They are the long-wavelength Fokker-Planck equations to second order, (105) and (119), that give the long-wavelength component of the distribution functions; the quasineutrality equation up to second-order (122), (130), and (131), that determines the first and second-order pieces of the long-wavelength poloidal electric field; and the transport equations for density (134) and energy (141). The first-order pieces of the short-wavelength components of the distribution functions and electrostatic potential appear in (119), and we give the equations for them in (109) and (110).

Carrying the expansion to second order in ϵ at long wavelengths also clarifies the issues with the radial electric field raised in references [12, 18, 19, 20, 21, 22], pointed out at the beginning of this introduction. Along with the derivation of the equations we give an explicit proof of the indeterminacy of the radial electric

field, showing that it cannot be found from the long-wavelength gyrokinetic Fokker-Planck and quasineutrality equations correct to second order. This property, known as intrinsic ambipolarity, was first proven for neoclassical transport in [30, 31] and it was shown to hold for turbulent tokamaks in [18] using the identical cancellation of the toroidal angular momentum conservation equation to the order of interest. The intrinsic ambipolarity of purely turbulent particle fluxes was shown to hold in [32], even electromagnetically and in general magnetic geometry (that is why the long-wavelength radial electric field in non-quasisymmetric stellarators is determined from neoclassical theory). This is, however, the first direct, explicit, and general proof for turbulent tokamaks. Instead of resorting to the toroidal angular momentum equation, we write the long-wavelength equations order by order and show that they can be solved for any radial electric field, leaving it undetermined. Those readers who are familiar with the Chapman-Enskog results on the derivation of fluid equations from kinetic theory (see the classical monograph [33]) will find that the approach that we adopt at some stages of the proof is very similar. The analogy becomes especially clear in Section 5.2. In previous sections the long-wavelength Fokker-Planck and quasineutrality equations have been derived up to second order. In Section 5.2 we inspect the second-order piece of the long-wavelength Fokker-Planck equation and learn that it possesses solvability conditions, i.e. the existence of solutions of this equation imposes constraints on lowest-order quantities. These constraints are transport equations for particle and energy density. The way of obtaining them and of showing that we have actually found all the solvability conditions are the aspects particularly reminiscent of the Chapman-Enskog techniques. Nevertheless, we have written the paper in a self-contained fashion and no prior knowledge of the Chapman-Enskog theory is assumed.

The rest of the paper is organized as follows. In Section 2 we introduce the gyrokinetic formulation and the essential results and notation from [16] that will be needed here. An important element of our derivation is the scale separation between the turbulent short-wavelength fluctuations and the equilibrium long-wavelength profiles. In Section 2 we also discuss the implications of this scale separation and formalize the notion of “taking the long-wavelength limit of gyrokinetics”. The most laborious part of this work corresponds to explicitly taking the long-wavelength limit of the gyrokinetic system of equations in tokamak geometry by employing the results of [16]. In Section 3 we do it for the Fokker-Planck equation and in Section 4 for the quasineutrality equation. Reaching the final expressions for the long-wavelength limit of the gyrokinetic system to second order involves enormous amounts of algebra, and in order to ease a first reading of the paper the most cumbersome parts of the calculation have been collected in the appendices. Using the results of Sections 3 and 4 we prove in Section 5 that the long-wavelength tokamak radial electric field is not determined by second-order Fokker-Planck and quasineutrality equations. A complete proof requires computing the solvability conditions imposed by the second-order long-wavelength Fokker-Planck equation, contained in Subsection 5.2. These conditions are transport equations for particle and energy densities, as mentioned above. With these transport equations, we

show in Section 5.3 that the well-known neoclassical intrinsic ambipolarity property of the tokamak is not broken by the turbulent terms that are specific to gyrokinetics, that is, the radial electric field is left undetermined by a gyrokinetic system of equations correct to second order in ϵ . Section 6 is devoted to a discussion of the results and the conclusions.

2. Second-order electrostatic gyrokinetics

In this section we state and justify the assumptions of the theory, and we summarize the results from reference [16] that will be needed.

2.1. Kinetic description of a plasma in a static magnetic field

The kinetic description of a plasma in the electrostatic approximation involves the Fokker-Planck equation for each species σ ,

$$\partial_t f_\sigma + \mathbf{v} \cdot \nabla_{\mathbf{r}} f_\sigma + \frac{Z_\sigma e}{m_\sigma} (-\nabla_{\mathbf{r}} \varphi + c^{-1} \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\sigma = \sum_{\sigma'} C_{\sigma\sigma'}[f_\sigma, f_{\sigma'}](\mathbf{r}, \mathbf{v}), \quad (1)$$

and Poisson's equation,

$$\nabla_{\mathbf{r}}^2 \varphi(\mathbf{r}, t) = -4\pi e \sum_{\sigma} Z_{\sigma} \int f_{\sigma}(\mathbf{r}, \mathbf{v}, t) d^3 v. \quad (2)$$

Here c is the speed of light, e the charge of the proton, $\varphi(\mathbf{r}, t)$ the electrostatic potential, $\mathbf{B}(\mathbf{r}) = \nabla_{\mathbf{r}} \times \mathbf{A}(\mathbf{r})$ a time-independent magnetic field, $f_{\sigma}(\mathbf{r}, \mathbf{v}, t)$ the phase-space probability distribution, and $Z_{\sigma}e$ and m_{σ} are the charge and the mass of species σ . We recall that the Landau collision operator between species σ and σ' reads

$$C_{\sigma\sigma'}[f_{\sigma}, f_{\sigma'}](\mathbf{r}, \mathbf{v}) = \frac{\gamma_{\sigma\sigma'}}{m_{\sigma}} \nabla_{\mathbf{v}} \cdot \int \overset{\leftrightarrow}{\mathbf{W}}(\mathbf{v} - \mathbf{v}') \cdot \left(\frac{1}{m_{\sigma}} f_{\sigma'}(\mathbf{r}, \mathbf{v}', t) \nabla_{\mathbf{v}} f_{\sigma}(\mathbf{r}, \mathbf{v}, t) - \frac{1}{m_{\sigma'}} f_{\sigma}(\mathbf{r}, \mathbf{v}, t) \nabla_{\mathbf{v}'} f_{\sigma'}(\mathbf{r}, \mathbf{v}', t) \right) d^3 v', \quad (3)$$

where

$$\gamma_{\sigma\sigma'} := 2\pi Z_{\sigma}^2 Z_{\sigma'}^2 e^4 \ln \Lambda, \quad (4)$$

$$\overset{\leftrightarrow}{\mathbf{W}}(\mathbf{w}) := \frac{|\mathbf{w}|^2 \overset{\leftrightarrow}{\mathbf{I}} - \mathbf{w} \mathbf{w}}{|\mathbf{w}|^3}, \quad (5)$$

$\ln \Lambda$ is the Coulomb logarithm, and $\overset{\leftrightarrow}{\mathbf{I}}$ is the identity matrix. A direct check shows that the Fokker-Planck equation can also be written as

$$\partial_t f_{\sigma} + \{f_{\sigma}, H_{\sigma}\}_{\mathbf{X}} = \sum_{\sigma'} C_{\sigma\sigma'}[f_{\sigma}, f_{\sigma'}](\mathbf{X}), \quad (6)$$

where we designate by $\mathbf{X} \equiv (\mathbf{r}, \mathbf{v})$ a set of euclidean coordinates in phase-space,

$$H_\sigma(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2}m_\sigma \mathbf{v}^2 + Z_\sigma e\varphi(\mathbf{r}, t) \quad (7)$$

is the Hamiltonian of species σ , and the Poisson bracket of two functions on phase space, $g_1(\mathbf{r}, \mathbf{v})$, $g_2(\mathbf{r}, \mathbf{v})$, is

$$\begin{aligned} \{g_1, g_2\}_{\mathbf{X}} &= \frac{1}{m_\sigma} (\nabla_{\mathbf{r}} g_1 \cdot \nabla_{\mathbf{v}} g_2 - \nabla_{\mathbf{v}} g_1 \cdot \nabla_{\mathbf{r}} g_2) \\ &+ \frac{Z_\sigma e}{m_\sigma^2 c} \mathbf{B} \cdot (\nabla_{\mathbf{v}} g_1 \times \nabla_{\mathbf{v}} g_2). \end{aligned} \quad (8)$$

2.2. Dimensionless variables

In most of what follows we find it convenient to work with non-dimensionalized variables [16]. The species-independent normalization

$$\begin{aligned} \underline{t} &= \frac{c_s t}{L}, \quad \underline{\mathbf{r}} = \frac{\mathbf{r}}{L}, \quad \underline{\mathbf{A}} = \frac{\mathbf{A}}{B_0 L}, \quad \underline{\varphi} = \frac{e\varphi}{\epsilon_s T_{e0}}, \\ \underline{H}_\sigma &= \frac{H_\sigma}{T_{e0}}, \quad \underline{n}_\sigma = \frac{n_\sigma}{n_{e0}}, \quad \underline{T}_\sigma = \frac{T_\sigma}{T_{e0}}, \end{aligned} \quad (9)$$

is employed for time, space, vector potential, electrostatic potential, Hamiltonian, particle density, and temperature; and the species-dependent normalization

$$\underline{\mathbf{v}}_\sigma = \frac{\mathbf{v}_\sigma}{v_{t\sigma}}, \quad \underline{f}_\sigma = \frac{v_{t\sigma}^3}{n_{e0}} f_\sigma, \quad (10)$$

for velocities and distribution functions. In the previous expressions $L \sim |\nabla_{\mathbf{r}} \ln |\mathbf{B}||^{-1}$ is the typical length of variation of the magnetic field, B_0 a typical value of the magnetic field strength, $c_s = \sqrt{T_{e0}/m_i}$ the sound speed, T_{e0} a typical electron temperature, n_{e0} a typical electron density, and m_i the mass of the dominant ion species, that we assume singly charged. Finally, $v_{t\sigma}$ is the thermal speed of species σ , $\epsilon_s = \rho_s/L$, where $\rho_s = c_s/\Omega_i$ is a characteristic sound gyroradius, and $\Omega_i = eB_0/(m_i c)$ is a characteristic ion gyrofrequency. We take $v_{t\sigma} = \sqrt{T_{e0}/m_\sigma}$ as the expression for the typical thermal speed, i.e. we assume that T_{e0} , the characteristic temperature of electrons, is also the characteristic temperature for all species. This assumption is justified when the time between collisions is shorter than the transport time scale, leading to thermal equilibration between species. The normalization of the electrostatic potential might seem strange at this point but it will be explained in the next subsection.

The natural, species-independent expansion parameter in gyrokinetic theory is ϵ_s . Many expressions, however, are more conveniently written in terms of the species-dependent parameter $\epsilon_\sigma = \rho_\sigma/L$, where $\rho_\sigma = v_{t\sigma}/\Omega_\sigma$ is a characteristic gyroradius of species σ and $\Omega_\sigma = Z_\sigma e B_0/(m_\sigma c)$ a characteristic gyrofrequency. Observe that the relation between ϵ_σ and ϵ_s is $\epsilon_s = \lambda_\sigma \epsilon_\sigma$, with

$$\lambda_\sigma = \frac{\rho_s}{\rho_\sigma} = Z_\sigma \sqrt{\frac{m_i}{m_\sigma}}. \quad (11)$$

In dimensionless variables, the Fokker-Planck equation (1) becomes

$$\partial_t \underline{f}_\sigma + \tau_\sigma \{ \underline{f}_\sigma, \underline{H}_\sigma \}_{\underline{\mathbf{X}}} = \tau_\sigma \sum_{\sigma'} \underline{C}_{\sigma\sigma'} [\underline{f}_\sigma, \underline{f}_{\sigma'}](\underline{\mathbf{r}}, \underline{\mathbf{v}}), \quad (12)$$

where

$$\tau_\sigma = \frac{v_{t\sigma}}{c_s} = \sqrt{\frac{m_i}{m_\sigma}}, \quad (13)$$

and the Poisson bracket of two functions $g_1(\underline{\mathbf{r}}, \underline{\mathbf{v}})$, $g_2(\underline{\mathbf{r}}, \underline{\mathbf{v}})$ (we no longer write the subindex σ in $\underline{\mathbf{v}}_\sigma$) is defined by

$$\begin{aligned} \{g_1, g_2\}_{\underline{\mathbf{X}}} &= (\nabla_{\underline{\mathbf{r}}} g_1 \cdot \nabla_{\underline{\mathbf{v}}} g_2 - \nabla_{\underline{\mathbf{v}}} g_1 \cdot \nabla_{\underline{\mathbf{r}}} g_2) \\ &\quad + \frac{1}{\epsilon_\sigma} \underline{\mathbf{B}} \cdot (\nabla_{\underline{\mathbf{v}}} g_1 \times \nabla_{\underline{\mathbf{v}}} g_2). \end{aligned} \quad (14)$$

Here $\underline{\mathbf{X}} \equiv (\underline{\mathbf{r}}, \underline{\mathbf{v}})$ are the dimensionless cartesian coordinates. The normalized collision operator is

$$\begin{aligned} \underline{C}_{\sigma\sigma'} [\underline{f}_\sigma, \underline{f}_{\sigma'}](\underline{\mathbf{r}}, \underline{\mathbf{v}}) &= \\ &\underline{\gamma}_{\sigma\sigma'} \nabla_{\underline{\mathbf{v}}} \cdot \int \overset{\leftrightarrow}{\mathbf{W}} (\tau_\sigma \underline{\mathbf{v}} - \tau_{\sigma'} \underline{\mathbf{v}}') \cdot \left(\tau_\sigma \underline{f}_{\sigma'}(\underline{\mathbf{r}}, \underline{\mathbf{v}}', \underline{t}) \nabla_{\underline{\mathbf{v}}} \underline{f}_\sigma(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) \right. \\ &\quad \left. - \tau_{\sigma'} \underline{f}_\sigma(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) \nabla_{\underline{\mathbf{v}}} \underline{f}_{\sigma'}(\underline{\mathbf{r}}, \underline{\mathbf{v}}', \underline{t}) \right) d^3 \underline{v}', \end{aligned} \quad (15)$$

with

$$\underline{\gamma}_{\sigma\sigma'} := \frac{2\pi Z_\sigma^2 Z_{\sigma'}^2 n_{e0} e^4 L}{T_{e0}^2} \ln \Lambda. \quad (16)$$

Note in passing that $\underline{\gamma}_{\sigma\sigma'}$ is the usual collisionality parameter $\nu_{*\sigma\sigma'}$ up to a factor of order unity. We use the following definition of $\nu_{*\sigma\sigma'}$:

$$\nu_{*\sigma\sigma'} := L \nu_{\sigma\sigma'} / v_{t\sigma}, \quad (17)$$

where the collision frequency is

$$\nu_{\sigma\sigma'} := \frac{4\sqrt{2\pi}}{3} \frac{Z_\sigma^2 Z_{\sigma'}^2 n_{e0} e^4}{m_\sigma^{1/2} T_\sigma^{3/2}} \ln \Lambda, \quad (18)$$

which coincides with Braginskii's definition [34] for $\sigma = e$ and $\sigma' = i$.

As for equation (2),

$$\frac{\epsilon_s \lambda_{De}^2}{L^2} \nabla_{\underline{\mathbf{r}}}^2 \varphi(\underline{\mathbf{r}}, \underline{t}) = - \sum_{\sigma} Z_\sigma \int \underline{f}_\sigma(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) d^3 \underline{v}, \quad (19)$$

where

$$\lambda_{De} = \sqrt{\frac{T_{e0}}{4\pi e^2 n_{e0}}} \quad (20)$$

is the electron Debye length. We assume that the Debye length is sufficiently small that we can neglect the left-hand side of (19), so quasineutrality

$$\sum_{\sigma} Z_\sigma \int \underline{f}_\sigma(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) d^3 \underline{v} = 0 \quad (21)$$

holds.

2.3. Gyrokinetic ordering and separation of scales

In strongly magnetized plasmas a small quantity, $\epsilon_\sigma = \rho_\sigma/L \ll 1$, naturally arises for each species. The smallness of ϵ_σ implies that two very different length scales exist: the gyroradius scale and the macroscopic scale. Also, strong magnetization makes the time scale associated to the gyromotion around a field line, Ω_σ^{-1} , very small compared to microturbulence time scales. It is therefore justified to try to average over the irrelevant gyromotion without losing non-zero gyroradius effects. Gyrokinetics is the theory that results from averaging over the gyromotion when the parameter ϵ_σ (or more precisely ϵ_s) is small. We assume that $\frac{\gamma_{\sigma\sigma'}}{\epsilon_s} \sim \lambda_\sigma \sim \tau_\sigma \sim 1$ for all σ, σ' . That is, the only formal expansion parameter is ϵ_s . This is a maximal expansion in the sense that the different physically reasonable and customary subsidiary expansions (such as expansions in mass ratios) are contained in our results and could be eventually performed in order to simplify the equations.

As most gyrokinetic derivations, this article relies on a set of *ansätze* about scale separation and ordering that we proceed to explain.

First we define a transport or coarse-grain average, that for a given function extracts the axisymmetric component (recall that our aim is to fully work out the axisymmetric case) corresponding to long wavelengths and small frequencies. Let $\{\psi, \Theta, \zeta\}$ be a set of flux coordinates, where ψ is the poloidal magnetic flux, Θ is the poloidal angle, and ζ is the toroidal angle. A working definition of this averaging operation can be given by

$$\langle \dots \rangle_T = \frac{1}{2\pi \Delta t \Delta \psi \Delta \Theta} \int_{\Delta t} dt \int_{\Delta \psi} d\psi \int_{\Delta \Theta} d\Theta \int_0^{2\pi} d\zeta (\dots), \quad (22)$$

where $\epsilon_s \ll \Delta \psi / \psi \ll 1$, $\epsilon_s \ll \Delta \Theta \ll 1$, and $L/c_s \ll \Delta t \ll \tau_E$. Here $\tau_E \sim \epsilon_s^{-2} L/c_s$ is the transport time scale. For any function $g(\mathbf{r}, t)$, we define

$$\begin{aligned} g^{\text{lw}} &:= \langle g \rangle_T \\ g^{\text{sw}} &:= g - g^{\text{lw}}. \end{aligned} \quad (23)$$

The following obvious properties will be repeatedly employed:

$$\begin{aligned} [g^{\text{lw}}]^{\text{lw}} &= g^{\text{lw}}, \\ [g^{\text{sw}}]^{\text{lw}} &= 0, \\ [gh]^{\text{lw}} &= g^{\text{lw}} h^{\text{lw}} + [g^{\text{sw}} h^{\text{sw}}]^{\text{lw}}, \end{aligned} \quad (24)$$

for any two functions $g(\mathbf{r}, t)$ and $h(\mathbf{r}, t)$. We decompose the fields of our theory using the coarse-grain average:

$$\begin{aligned} f_\sigma &= f_\sigma^{\text{lw}} + f_\sigma^{\text{sw}}, \\ \varphi &= \varphi^{\text{lw}} + \varphi^{\text{sw}}. \end{aligned} \quad (25)$$

An *ansatz* is made about the relative size of the long-wavelength and the short-wavelength components. The long-wavelength component of the distribution function is assumed to be larger than the short-wavelength piece by a factor of $\epsilon_s^{-1} \gg 1$; the

long-wavelength piece of the potential is itself comparable to the kinetic energy of the particles and its short-wavelength component is also small in ϵ_s . Summarizing,

$$\begin{aligned}\frac{v_{t\sigma}^3 f_{\sigma}^{\text{sw}}}{n_{e0}} &\sim \frac{Z_{\sigma} e \varphi^{\text{sw}}}{m_{\sigma} v_{t\sigma}^2} \sim \epsilon_s, \\ \frac{v_{t\sigma}^3 f_{\sigma}^{\text{lw}}}{n_{e0}} &\sim \frac{Z_{\sigma} e \varphi^{\text{lw}}}{m_{\sigma} v_{t\sigma}^2} \sim 1.\end{aligned}\quad (26)$$

We need also an *ansatz* about the size of the space and time derivatives of the long- and short-wavelength components of our fields. The long-wavelength components f_{σ}^{lw} and φ^{lw} are characterized by large spatial scales, of the order of the macroscopic scale L , and long time scales, of the order of the transport time scale, $\tau_E := L/(c_s \epsilon_s^2)$, i.e.

$$\begin{aligned}\nabla_{\mathbf{r}} \ln f_{\sigma}^{\text{lw}}, \nabla_{\mathbf{r}} \ln \varphi^{\text{lw}} &\sim 1/L, \\ \partial_t \ln f_{\sigma}^{\text{lw}}, \partial_t \ln \varphi^{\text{lw}} &\sim \epsilon_s^2 c_s / L.\end{aligned}\quad (27)$$

The short-wavelength components f_{σ}^{sw} and φ^{sw} have perpendicular wavelengths of the order of the sound gyroradius, and short time scales, of the order of the turbulent correlation time. The parallel correlation length of the short-wavelength component is much longer than its characteristic perpendicular wavelength, and it is comparable to the size of the machine. In short, f_{σ}^{sw} and φ^{sw} are characterized by

$$\begin{aligned}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \ln f_{\sigma}^{\text{sw}}, \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \ln \varphi^{\text{sw}} &\sim 1/L, \\ \nabla_{\mathbf{r}_{\perp}} \ln f_{\sigma}^{\text{sw}}, \nabla_{\mathbf{r}_{\perp}} \ln \varphi^{\text{sw}} &\sim 1/\rho_s, \\ \partial_t \ln f_{\sigma}^{\text{sw}}, \partial_t \ln \varphi^{\text{sw}} &\sim c_s / L.\end{aligned}\quad (28)$$

The magnetic field only contains long-wavelength components,

$$\nabla_{\mathbf{r}} \ln |\mathbf{B}| \sim 1/L. \quad (29)$$

The above assumptions make the elimination of the gyrophase order by order in ϵ_s possible and the resulting equations consistent. These assumptions are based on experimental and theoretical evidence. In experiments it has been possible to confirm that the characteristic correlation length of the turbulence is of the order of and scales with the ion gyroradius [35]. The same measurements showed that the size of the turbulent fluctuations scales with the ion gyroradius. The characteristic length of the turbulent eddies and the size of the fluctuations are related to each other by the background gradient. An eddy of length $\ell_{\perp} \sim \rho_s$ mixes the plasma contained within it. In the presence of a gradient this eddy will lead to fluctuations on top of the background density of order $\delta n_e \sim \ell_{\perp} |\nabla n_e| \sim \epsilon_s n_e \ll n_e$.

In addition to the experimental measurements, there exist strong theoretical arguments in favor of the assumptions above. The equations obtained using these assumptions lead to a nonlinear system of gyrokinetic equations for the fluctuations. These equations can be implemented in numerical simulations that encompass several ion gyroradii, as is done in [3, 4, 5, 7]. These simulations converge for numerical domains that are sufficiently large to contain the largest turbulent eddies. The model is

consistent if the domain size is only several gyroradii across, proving that for sufficiently small gyroradius the turbulence eddies will scale with the gyroradius. The flux tube simulations converge, and the characteristic size of the turbulent eddies is indeed of the order of the ion gyroradius. In [36] the fluctuation spectrum of the turbulence is studied by varying different parameters. The final result is that the spectrum peaks around wavelengths proportional to the ion gyroradius, and although the constant of proportionality depends on the density and temperature gradients and the magnetic field characteristics, it is of order unity. The size of the fluctuations is also of order ϵ_s .

Observe that in dimensionless variables the short-wavelength electrostatic potential and distribution functions satisfy

$$\begin{aligned}
\underline{\varphi}^{\text{sw}}(\underline{\mathbf{r}}, \underline{t}) &\sim 1, \\
\underline{f}_\sigma^{\text{sw}}(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) &\sim \epsilon_s, \\
\hat{\mathbf{b}}(\underline{\mathbf{r}}) \cdot \nabla_{\underline{\mathbf{r}}} \underline{\varphi}^{\text{sw}}(\underline{\mathbf{r}}, \underline{t}) &\sim 1, \\
\hat{\mathbf{b}}(\underline{\mathbf{r}}) \cdot \nabla_{\underline{\mathbf{r}}} \underline{f}_\sigma^{\text{sw}}(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) &\sim \epsilon_s, \\
\nabla_{\underline{\mathbf{r}}_\perp} \underline{\varphi}^{\text{sw}}(\underline{\mathbf{r}}, \underline{t}) &\sim 1/\epsilon_s, \\
\nabla_{\underline{\mathbf{r}}_\perp} \underline{f}_\sigma^{\text{sw}}(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) &\sim 1, \\
\partial_{\underline{t}} \underline{\varphi}^{\text{sw}}(\underline{\mathbf{r}}, \underline{t}) &\sim 1, \\
\partial_{\underline{t}} \underline{f}_\sigma^{\text{sw}}(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) &\sim \epsilon_s.
\end{aligned} \tag{30}$$

The normalized functions $\underline{\varphi}^{\text{sw}}$ and $\underline{f}_\sigma^{\text{sw}}$ are of different size due to our choice of dimensionless variables, consistent with [16] (see (9) and (10)).

As for the long-wavelength components,

$$\begin{aligned}
\underline{\varphi}^{\text{lw}}(\underline{\mathbf{r}}, \underline{t}) &\sim 1/\epsilon_s, \\
\underline{f}_\sigma^{\text{lw}}(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) &\sim 1, \\
\nabla_{\underline{\mathbf{r}}} \underline{\varphi}^{\text{lw}}(\underline{\mathbf{r}}, \underline{t}) &\sim 1/\epsilon_s, \\
\nabla_{\underline{\mathbf{r}}} \underline{f}_\sigma^{\text{lw}}(\underline{\mathbf{r}}, \underline{t}) &\sim 1, \\
\partial_{\underline{t}} \underline{\varphi}^{\text{lw}}(\underline{\mathbf{r}}, \underline{t}) &\sim \epsilon_s, \\
\partial_{\underline{t}} \underline{f}_\sigma^{\text{lw}}(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) &\sim \epsilon_s^2.
\end{aligned} \tag{31}$$

The following convention is adopted when we expand $\underline{\varphi}^{\text{lw}}(\underline{\mathbf{r}}, \underline{t})$ in powers of ϵ_s :

$$\underline{\varphi}^{\text{lw}}(\underline{\mathbf{r}}, \underline{t}) := \frac{1}{\epsilon_s} \underline{\varphi}_0^{\text{lw}}(\underline{\mathbf{r}}, \underline{t}) + \underline{\varphi}_1^{\text{lw}}(\underline{\mathbf{r}}, \underline{t}) + \epsilon_s \underline{\varphi}_2^{\text{lw}}(\underline{\mathbf{r}}, \underline{t}) + O(\epsilon_s^2). \tag{32}$$

Similarly,

$$\underline{\varphi}^{\text{sw}}(\underline{\mathbf{r}}, \underline{t}) := \underline{\varphi}_1^{\text{sw}}(\underline{\mathbf{r}}, \underline{t}) + \epsilon_s \underline{\varphi}_2^{\text{sw}}(\underline{\mathbf{r}}, \underline{t}) + O(\epsilon_s^2). \tag{33}$$

From now on we do not underline variables but assume that we are working with the dimensionless ones unless otherwise stated.

2.4. Gyrokinetic expansion to second order

The complete calculation of the gyrokinetic system of equations to second order is given for the first time in reference [16] in the phase-space Lagrangian formalism. The latter was applied to the problem of guiding-center motion by Littlejohn [37] and has been used extensively in modern formulations of gyrokinetics [15]. In reference [16] we perform a change of variables in (12) and (21) that decouples the fast degree of freedom (the gyrophase) from the slow ones in the absence of collisions. This decoupling is achieved by eliminating the dependence on the gyration order by order in ϵ_σ . Let us denote the transformation \S from the new phase-space coordinates $\mathbf{Z} \equiv \{\mathbf{R}, u, \mu, \theta\}$ to the euclidean ones $\mathbf{X} \equiv \{\mathbf{r}, \mathbf{v}\}$ by \mathcal{T}_σ ,

$$(\mathbf{r}, \mathbf{v}) = \mathcal{T}_\sigma(\mathbf{R}, u, \mu, \theta, t). \quad (34)$$

The transformation is, in general, explicitly time-dependent and is expressed as a power series in ϵ_σ . Here \mathbf{R} is the position of the gyrocenter, and u , μ , and θ are deformations of the parallel velocity, magnetic moment, and gyrophase. We recall that in [16] the gyrokinetic transformation is written as the composition of two transformations. First, the *non-perturbative transformation*, $(\mathbf{r}, \mathbf{v}) = \mathcal{T}_{NP,\sigma}(\mathbf{Z}_g) \equiv \mathcal{T}_{NP,\sigma}(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g)$,

$$\begin{aligned} \mathbf{r} &= \mathbf{R}_g + \epsilon_\sigma \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g), \\ \mathbf{v} &= v_{||g} \hat{\mathbf{b}}(\mathbf{R}_g) + \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g) \times \mathbf{B}(\mathbf{R}_g), \end{aligned} \quad (35)$$

with the gyroradius vector defined as

$$\boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g) = -\sqrt{\frac{2\mu_g}{B(\mathbf{R}_g)}} [\sin \theta_g \hat{\mathbf{e}}_1(\mathbf{R}_g) - \cos \theta_g \hat{\mathbf{e}}_2(\mathbf{R}_g)]. \quad (36)$$

The unit vectors $\hat{\mathbf{e}}_1(\mathbf{r})$ and $\hat{\mathbf{e}}_2(\mathbf{r})$ are orthogonal to each other and to $\hat{\mathbf{b}} = \mathbf{B}/B$, and satisfy $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$ at every location \mathbf{r} . Second, the *perturbative transformation*

$$(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g) = \mathcal{T}_{P,\sigma}(\mathbf{R}, u, \mu, \theta, t), \quad (37)$$

that we express as

$$\begin{aligned} \mathbf{R}_g &= \mathbf{R} + \sum_{i=1}^n \epsilon_\sigma^{i+1} \mathbf{R}_{i+1}, \\ v_{||g} &= u + \sum_{i=1}^n \epsilon_\sigma^i u_i, \\ \mu_g &= \mu + \sum_{i=1}^n \epsilon_\sigma^i \mu_i, \\ \theta_g &= \theta + \sum_{i=1}^n \epsilon_\sigma^i \theta_i. \end{aligned} \quad (38)$$

The gyrokinetic transformation is

$$\mathcal{T}_\sigma = \mathcal{T}_{NP,\sigma} \mathcal{T}_{P,\sigma}. \quad (39)$$

\S We warn the reader that we call \mathcal{T}_σ to the transformation that is often called \mathcal{T}_σ^{-1} in the literature.

At this point we need to mention that the derivation of \mathcal{T}_σ in [16] assumed that the electrostatic potential had only a short-wavelength component, i.e. we assumed $\varphi = \varphi^{\text{sw}}$ and $\varphi^{\text{lw}} = 0$. Since φ^{sw} is small in ϵ_s , this assumption lead to normalizing the electrostatic potential with $\epsilon_s T_{e0}/e$. It is easy to relax the assumption in [16] that $\varphi = \varphi^{\text{sw}}$ and include φ^{lw} . In equations (68) and (69) of [16] we display the phase-space Lagrangian after the non-perturbative transformation. The Hamiltonian is given by

$$H = H^{(0)} + \epsilon_\sigma H^{(1)}, \quad (40)$$

with

$$H^{(0)} = \frac{1}{2} v_{\parallel g}^2 + \mu_g B(\mathbf{R}_g) \quad (41)$$

and

$$H^{(1)} = Z_\sigma \lambda_\sigma \varphi^{\text{sw}}(\mathbf{R}_g + \epsilon_\sigma \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g), t). \quad (42)$$

Using this expression, it is possible to obtain the perturbative change of variables $\mathcal{T}_{P,\sigma}$ by expanding in ϵ_σ . To do so, $H^{(1)}$ must be of order unity, and if instead of $\varphi = \varphi^{\text{sw}}$ we have a long wavelength piece $\varphi^{\text{lw}} \sim 1/\epsilon_s \gg 1$, it would seem that the condition $H^{(1)} \sim 1$ is not satisfied. Fortunately, it is possible to redefine $H^{(0)}$ and $H^{(1)}$ so that the expansion can be performed. The new Hamiltonian is given by

$$H^{(0)} = \frac{1}{2} v_{\parallel g}^2 + \mu_g B(\mathbf{R}_g) + Z_\sigma \lambda_\sigma \epsilon_\sigma \langle \phi_\sigma \rangle(\mathbf{R}_g, \mu_g, t) \quad (43)$$

and

$$H^{(1)} = Z_\sigma \lambda_\sigma \tilde{\phi}_\sigma(\mathbf{R}_g, \mu_g, \theta_g, t), \quad (44)$$

where the function ϕ_σ is defined as

$$\phi_\sigma(\mathbf{R}_g, \mu_g, \theta_g, t) := \varphi(\mathbf{R}_g + \epsilon_\sigma \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g), t). \quad (45)$$

From it we can calculate

$$\tilde{\phi}_\sigma(\mathbf{R}_g, \mu_g, \theta_g, t) := \phi_\sigma(\mathbf{R}_g, \mu_g, \theta_g, t) - \langle \phi_\sigma \rangle(\mathbf{R}_g, \mu_g, t) \quad (46)$$

and

$$\langle \phi_\sigma \rangle(\mathbf{R}_g, \mu_g, t) := \frac{1}{2\pi} \int_0^{2\pi} \phi_\sigma(\mathbf{R}_g, \mu_g, \theta_g, t) d\theta_g. \quad (47)$$

Here $\langle \cdot \rangle$ stands for the average over the gyrophase. We now prove that $H^{(1)}$ is indeed of order unity. From the ordering and scale separation assumptions on φ , equations (30) and (31), we obtain that the turbulent component of ϕ_σ is $O(1)$, i.e.

$$\begin{aligned} \phi_\sigma^{\text{sw}} &= \phi_{\sigma 1}^{\text{sw}} + O(\epsilon_s), \\ \tilde{\phi}_\sigma^{\text{sw}} &= \tilde{\phi}_{\sigma 1}^{\text{sw}} + O(\epsilon_s). \end{aligned} \quad (48)$$

For the long wavelength piece ϕ_σ^{lw} we use that it is possible to Taylor expand around $\mathbf{r} = \mathbf{R}$ to find

$$\begin{aligned} \langle \phi_\sigma^{\text{lw}} \rangle(\mathbf{R}_g, \mu_g, t) &= \frac{1}{\epsilon_s} \varphi_0(\mathbf{R}_g, t) + \varphi_1^{\text{lw}}(\mathbf{R}_g, t) \\ &+ \epsilon_s \left(\frac{\mu_g}{2\lambda_\sigma^2 B(\mathbf{R}_g)} (\hat{\mathbf{I}} - \hat{\mathbf{b}}(\mathbf{R}_g) \hat{\mathbf{b}}(\mathbf{R}_g)) : \nabla_{\mathbf{R}_g} \nabla_{\mathbf{R}_g} \varphi_0(\mathbf{R}_g, t) + \varphi_2^{\text{lw}}(\mathbf{R}_g, t) \right) \\ &+ O(\epsilon_s^2) \end{aligned} \quad (49)$$

and

$$\tilde{\phi}_\sigma^{\text{lw}}(\mathbf{R}_g, \mu_g, \theta_g, t) = \frac{1}{\lambda_\sigma} \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g) \cdot \nabla_{\mathbf{R}_g} \varphi_0(\mathbf{R}_g, t) + O(\epsilon_s), \quad (50)$$

giving $\tilde{\phi}_\sigma^{\text{lw}} = O(1)$ as expected. We have expanded up to first order in ϵ_s in (49) because it will be needed later in this paper. Our double-dot convention for arbitrary matrix $\overset{\leftrightarrow}{\mathbf{M}}$ is $\mathbf{u} \mathbf{v} : \overset{\leftrightarrow}{\mathbf{M}} = \mathbf{v} \cdot \overset{\leftrightarrow}{\mathbf{M}} \cdot \mathbf{u}$. The authors of references [38] and [39] already pointed out the usefulness of the separation of the electrostatic potential into a large gyrophase-independent piece and a small gyrophase-dependent one, and exploited it in their derivations.

We want to write the Fokker-Planck equation in gyrokinetic coordinates. Denote by \mathcal{T}_σ^* the pull-back transformation induced by \mathcal{T}_σ . Acting on a function $g(\mathbf{X}, t)$, $\mathcal{T}_\sigma^* g(\mathbf{Z}, t)$ is simply the function g written in the coordinates \mathbf{Z} , i.e.

$$\mathcal{T}_\sigma^* g(\mathbf{Z}, t) = g(\mathcal{T}_\sigma(\mathbf{Z}, t), t). \quad (51)$$

Now, defining $F_\sigma := \mathcal{T}_\sigma^* f_\sigma$, we transform (12) and get:

$$\partial_t F_\sigma + \tau_\sigma \{F_\sigma, \overline{H}_\sigma\}_{\mathbf{Z}} = \tau_\sigma \sum_{\sigma'} \mathcal{T}_\sigma^* C_{\sigma\sigma'} [\mathcal{T}_\sigma^{-1*} F_\sigma, \mathcal{T}_{\sigma'}^{-1*} F_{\sigma'}](\mathbf{Z}, t), \quad (52)$$

where \mathcal{T}_σ^{-1*} is the pull-back transformation that corresponds to \mathcal{T}_σ^{-1} , i.e. $\mathcal{T}_\sigma^{-1*} F_\sigma(\mathbf{X}, t) = F_\sigma(\mathcal{T}_\sigma^{-1}(\mathbf{X}, t), t)$, and the Poisson bracket in the new coordinates is expressed as

$$\begin{aligned} \{G_1, G_2\}_{\mathbf{Z}} &= \frac{1}{\epsilon_\sigma} (\partial_\mu G_1 \partial_\theta G_2 - \partial_\theta G_1 \partial_\mu G_2) \\ &+ \frac{1}{B_{||,\sigma}^*} \mathbf{B}_\sigma^* \cdot (\nabla_{\mathbf{R}}^* G_1 \partial_u G_2 - \partial_u G_1 \nabla_{\mathbf{R}}^* G_2) \\ &+ \frac{\epsilon_\sigma}{B_{||}^*} \nabla_{\mathbf{R}}^* G_1 \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}}^* G_2), \end{aligned} \quad (53)$$

with

$$\mathbf{B}_\sigma^*(\mathbf{R}, u, \mu) := \mathbf{B}(\mathbf{R}) + \epsilon_\sigma u \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}(\mathbf{R}) - \epsilon_\sigma^2 \mu \nabla_{\mathbf{R}} \times \mathbf{K}(\mathbf{R}), \quad (54)$$

$$\begin{aligned} B_{||,\sigma}^*(\mathbf{R}, u, \mu) &:= \mathbf{B}_\sigma^*(\mathbf{R}, u, \mu) \cdot \hat{\mathbf{b}}(\mathbf{R}) \\ &= B(\mathbf{R}) + \epsilon_\sigma u \hat{\mathbf{b}}(\mathbf{R}) \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}(\mathbf{R}) \\ &\quad - \epsilon_\sigma^2 \mu \hat{\mathbf{b}}(\mathbf{R}) \cdot \nabla_{\mathbf{R}} \times \mathbf{K}(\mathbf{R}), \end{aligned} \quad (55)$$

$$\nabla_{\mathbf{R}}^* := \nabla_{\mathbf{R}} - \mathbf{K}(\mathbf{R}) \partial_\theta, \quad (56)$$

and

$$\mathbf{K}(\mathbf{R}) = \frac{1}{2} \hat{\mathbf{b}}(\mathbf{R}) \hat{\mathbf{b}}(\mathbf{R}) \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}(\mathbf{R}) - \nabla_{\mathbf{R}} \hat{\mathbf{e}}_2(\mathbf{R}) \cdot \hat{\mathbf{e}}_1(\mathbf{R}). \quad (57)$$

The above expression (53) for the Poisson bracket is customary in the literature [15]. The gyrokinetic change of coordinates is not unique, in the sense that there are infinitely many transformations such that the gyromotion is decoupled from the rest of degrees of freedom and such that the coordinate μ is an adiabatic invariant. To make comparisons

with standard references in the literature easy, we have made use of this flexibility by choosing our change of variables so that the Poisson bracket takes the form (53).

The main achievement of [16] was the computation of the gyrokinetic Hamiltonian $\overline{H}_\sigma = \sum_{n=0}^{\infty} \epsilon_\sigma^n \overline{H}_\sigma^{(n)}$ to order ϵ_σ^2 . The result is:

$$\overline{H}_\sigma^{(0)} = \frac{1}{2}u^2 + \mu B, \quad (58)$$

$$\overline{H}_\sigma^{(1)} = Z_\sigma \lambda_\sigma \langle \phi_\sigma \rangle, \quad (59)$$

$$\overline{H}_\sigma^{(2)} = Z_\sigma^2 \lambda_\sigma^2 \Psi_{\phi,\sigma} + Z_\sigma \lambda_\sigma \Psi_{\phi B,\sigma} + \Psi_{B,\sigma}, \quad (60)$$

with

$$\begin{aligned} \Psi_{\phi,\sigma} &= \frac{1}{2B^2} \left\langle \nabla_{(\mathbf{R}_\perp/\epsilon_\sigma)} \tilde{\Phi}_\sigma \cdot \left(\hat{\mathbf{b}} \times \nabla_{(\mathbf{R}_\perp/\epsilon_\sigma)} \tilde{\Phi}_\sigma \right) \right\rangle \\ &\quad - \frac{1}{2B} \partial_\mu \langle \tilde{\phi}_\sigma^2 \rangle, \end{aligned} \quad (61)$$

$$\begin{aligned} \Psi_{\phi B,\sigma} &= -\frac{u}{B} \left\langle \left(\nabla_{(\mathbf{R}_\perp/\epsilon_\sigma)} \tilde{\Phi}_\sigma \times \hat{\mathbf{b}} \right) \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} \right\rangle \\ &\quad - \frac{\mu}{2B^2} \nabla_{\mathbf{R}} B \cdot \nabla_{(\mathbf{R}_\perp/\epsilon_\sigma)} \langle \phi_\sigma \rangle - \frac{1}{B} \nabla_{\mathbf{R}} B \cdot \langle \tilde{\phi}_\sigma \boldsymbol{\rho} \rangle \\ &\quad - \frac{1}{4B} \left\langle \nabla_{(\mathbf{R}_\perp/\epsilon_\sigma)} \tilde{\Phi}_\sigma \cdot \left[\boldsymbol{\rho} \boldsymbol{\rho} - (\boldsymbol{\rho} \times \hat{\mathbf{b}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}} B \right\rangle \\ &\quad - \frac{u^2}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \left\langle \partial_\mu \tilde{\phi}_\sigma \boldsymbol{\rho} \right\rangle - \frac{u^2}{2\mu B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \langle \tilde{\phi}_\sigma \boldsymbol{\rho} \rangle \\ &\quad + \frac{u}{4} \nabla_{\mathbf{R}} \hat{\mathbf{b}} : \left\langle \partial_\mu \tilde{\phi}_\sigma \left[\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}})\boldsymbol{\rho} \right] \right\rangle \\ &\quad + \frac{u}{4\mu} \nabla_{\mathbf{R}} \hat{\mathbf{b}} : \left\langle \tilde{\phi}_\sigma \left[\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}})\boldsymbol{\rho} \right] \right\rangle \end{aligned} \quad (62)$$

and

$$\begin{aligned} \Psi_{B,\sigma} &= -\frac{3u^2\mu}{2B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \\ &\quad + \frac{\mu^2}{4B} (\overset{\leftrightarrow}{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} B \cdot \hat{\mathbf{b}} \\ &\quad - \frac{3\mu^2}{4B^2} |\nabla_{\mathbf{R}_\perp} B|^2 + \frac{u^2\mu}{2B} \nabla_{\mathbf{R}} \hat{\mathbf{b}} : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \\ &\quad + \left(\frac{\mu^2}{8} - \frac{u^2\mu}{4B} \right) \nabla_{\mathbf{R}} \hat{\mathbf{b}} : (\nabla_{\mathbf{R}} \hat{\mathbf{b}})^{\text{T}} \\ &\quad - \left(\frac{3u^2\mu}{8B} + \frac{\mu^2}{16} \right) (\nabla_{\mathbf{R}} \cdot \hat{\mathbf{b}})^2 \\ &\quad + \left(\frac{3u^2\mu}{2B} - \frac{u^4}{2B^2} \right) |\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}|^2 \\ &\quad + \left(\frac{u^2\mu}{8B} - \frac{\mu^2}{16} \right) (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}})^2. \end{aligned} \quad (63)$$

Here $\overset{\leftrightarrow}{\mathbf{M}}^{\text{T}}$ is the transpose of an arbitrary matrix matrix $\overset{\leftrightarrow}{\mathbf{M}}$, the magnetic field quantities $\mathbf{B}(\mathbf{R})$, $\hat{\mathbf{b}}(\mathbf{R})$ and $B(\mathbf{R})$ are evaluated at \mathbf{R} instead of \mathbf{r} , the functions $\phi_\sigma(\mathbf{R}, \mu, \theta, t)$,

$\langle \phi_\sigma \rangle(\mathbf{R}, \mu, t)$ and $\tilde{\phi}_\sigma(\mathbf{R}, \mu, \theta, t)$ are evaluated at \mathbf{R} , μ and θ instead of \mathbf{R}_g , μ_g and θ_g , and

$$\tilde{\Phi}_\sigma(\mathbf{R}, \mu, \theta, t) := \int^\theta \tilde{\phi}_\sigma(\mathbf{R}, \mu, \theta', t) d\theta', \quad (64)$$

where the lower limit of the integral is chosen such that $\langle \tilde{\Phi}_\sigma \rangle = 0$. The second-order Hamiltonian is sufficient to obtain the long-wavelength component of the distribution function to order ϵ_σ^2 . To check this, the reader can follow the calculation in this article assuming that $\bar{H}_\sigma^{(n)}$ for $n > 2$ are known, and finding that these higher-order terms do not enter the final equations.

In gyrokinetic variables the quasineutrality equation reads

$$\sum_\sigma Z_\sigma \int |\det(J_\sigma)| F_\sigma \delta\left(\pi^{\mathbf{r}}\left(\mathcal{T}_\sigma(\mathbf{Z}, t)\right) - \mathbf{r}\right) d^6 Z = 0, \quad (65)$$

with $\pi^{\mathbf{r}}(\mathbf{r}, \mathbf{v}) := \mathbf{r}$, and the Jacobian of the transformation to order ϵ_σ^2 is

$$|\det(J_\sigma)| \equiv B_{||, \sigma}^*. \quad (66)$$

The expressions for the corrections \mathbf{R}_2 , u_1 , μ_1 , and θ_1 found in [16] are

$$\begin{aligned} \mathbf{R}_{\sigma,2} = & -\frac{2u}{B} \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot (\boldsymbol{\rho} \times \hat{\mathbf{b}}) - \frac{u}{B} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} \\ & - \frac{1}{8} \hat{\mathbf{b}} \left[\boldsymbol{\rho} \boldsymbol{\rho} - (\boldsymbol{\rho} \times \hat{\mathbf{b}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}) \right] : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \\ & - \frac{1}{2B} \boldsymbol{\rho} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B - \frac{Z_\sigma \lambda_\sigma}{B^2} \hat{\mathbf{b}} \times \nabla_{(\mathbf{R}_\perp / \epsilon_\sigma)} \tilde{\Phi}_\sigma, \end{aligned} \quad (67)$$

$$\begin{aligned} u_{\sigma,1} = & u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} \\ & - \frac{B}{4} \left[\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}})\boldsymbol{\rho} \right] : \nabla_{\mathbf{R}} \hat{\mathbf{b}}, \end{aligned} \quad (68)$$

$$\begin{aligned} \mu_{\sigma,1} = & -\frac{Z_\sigma \lambda_\sigma \tilde{\phi}_\sigma}{B} - \frac{u^2}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} \\ & + \frac{u}{4} \left[\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}})\boldsymbol{\rho} \right] : \nabla_{\mathbf{R}} \hat{\mathbf{b}}, \end{aligned} \quad (69)$$

$$\begin{aligned} \theta_{\sigma,1} = & \frac{Z_\sigma \lambda_\sigma}{B} \partial_\mu \tilde{\Phi}_\sigma + \frac{u^2}{2\mu B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot (\boldsymbol{\rho} \times \hat{\mathbf{b}}) \\ & + \frac{u}{8\mu} \left[\boldsymbol{\rho} \boldsymbol{\rho} - (\boldsymbol{\rho} \times \hat{\mathbf{b}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}) \right] : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \\ & + \frac{1}{B} (\boldsymbol{\rho} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} B. \end{aligned} \quad (70)$$

These corrections are needed to find the gyrokinetic quasineutrality equation to the order of interest. Although it might seem that we also need the next order corrections $\mathbf{R}_{\sigma,3}$, $u_{\sigma,2}$, $\mu_{\sigma,2}$, and $\theta_{\sigma,2}$, it will be shown that they do not contribute in the long-wavelength limit.

In the following sections we take the long-wavelength limit of (52) and (65) up to second-order in the expansion parameter.

3. Fokker-Planck equation at long wavelengths

The objective in this section is to take the long-wavelength limit of the gyrokinetic Fokker-Planck equation (52) up to second order in tokamak geometry. As a preliminary step we must write (52) order by order; for this we will expand F_σ as

$$F_\sigma = \sum_{n=0}^{\infty} \epsilon_\sigma^n F_{\sigma n} = \sum_{n=0}^{\infty} \epsilon_\sigma^n F_{\sigma n}^{\text{lw}} + \sum_{n=1}^{\infty} \epsilon_\sigma^n F_{\sigma n}^{\text{sw}}. \quad (71)$$

From the scale separation and ordering assumptions enumerated in Section 2 it follows that

$$\begin{aligned} F_{\sigma n} &\sim 1, \quad n \geq 0, \\ \hat{\mathbf{b}}(\mathbf{R}) \cdot \nabla_{\mathbf{R}} F_{\sigma n} &\sim 1, \quad n \geq 0. \end{aligned} \quad (72)$$

Also, the long-wavelength component of every $F_{\sigma n}$ must have perpendicular derivatives of order unity in normalized variables, i.e.

$$\nabla_{\mathbf{R}_\perp} F_{\sigma n}^{\text{lw}} \sim 1, \quad n \geq 0. \quad (73)$$

Finally, the zeroth-order distribution function must have an identically vanishing short-wavelength component, and the perpendicular gradient of the rest of the short-wavelength components is of order ϵ_σ^{-1} ,

$$\begin{aligned} F_{\sigma 0}^{\text{sw}} &\equiv 0, \\ \nabla_{\mathbf{R}_\perp} F_{\sigma n}^{\text{sw}} &\sim \epsilon_\sigma^{-1}, \quad n \geq 1. \end{aligned} \quad (74)$$

Then, one must use expression (53) for the Poisson bracket in gyrokinetic coordinates and the form of \overline{H}_σ given in equations (58), (59), and (60). With the help of Appendix A, writing (52) order by order is relatively straightforward. In addition to writing the equations order by order, we manipulate them to make them as close as possible to the results obtained in neoclassical theory [26, 27]. This form of the equations will be useful when we calculate the transport equations for particles and energy in subsection 5.2.

Recall that along this paper we assume $\gamma_{\sigma\sigma'} \sim \lambda_\sigma \sim \tau_\sigma \sim 1$ for all σ, σ' , so that the only formal expansion parameter is ϵ_s .

3.1. Long-wavelength Fokker-Planck equation to $O(\epsilon_\sigma^{-1})$

The coefficient of ϵ_σ^{-1} in (52) simply gives

$$-\tau_\sigma B \partial_\theta F_{\sigma 0} = 0, \quad (75)$$

implying that $F_{\sigma 0}$ is independent of θ .

3.2. Long-wavelength Fokker-Planck equation to $O(\epsilon_\sigma^0)$

Equation (52) to order ϵ_σ^0 involves the collision operator, which is written in coordinates $\mathbf{X} \equiv (\mathbf{r}, \mathbf{v})$. Therefore, either we transform the collision operator to gyrokinetic coordinates $\mathbf{Z} \equiv (\mathbf{R}, u, \mu, \theta)$ or transform the gyrokinetic distribution function to

coordinates \mathbf{X} . We choose the second option. To write order by order the collision operator we need to obtain certain coefficients of the Taylor expansion of the gyrokinetic change of coordinates \mathcal{T}_σ and its inverse, \mathcal{T}_σ^{-1} ,

$$\mathbf{X} = \mathcal{T}_\sigma(\mathbf{Z}, t) = \mathcal{T}_{\sigma,0}(\mathbf{Z}, t) + \epsilon_\sigma \mathcal{T}_{\sigma,1}(\mathbf{Z}, t) + O(\epsilon_\sigma^2), \quad (76)$$

$$\begin{aligned} \mathbf{Z} = \mathcal{T}_\sigma^{-1}(\mathbf{X}, t) &= \mathcal{T}_{\sigma,0}^{-1}(\mathbf{X}, t) + \epsilon_\sigma \mathcal{T}_{\sigma,1}^{-1}(\mathbf{X}, t) \\ &+ \epsilon_\sigma^2 \mathcal{T}_{\sigma,2}^{-1}(\mathbf{X}, t) + O(\epsilon_\sigma^3). \end{aligned} \quad (77)$$

In the present subsection we need $\mathcal{T}_{\sigma,0}$, the transformation \mathcal{T}_σ , equation (39), for $\epsilon_\sigma = 0$:

$$\mathcal{T}_{\sigma,0}(\mathbf{R}, u, \mu, \theta) = (\mathbf{R}, u \hat{\mathbf{b}}(\mathbf{R}) + \boldsymbol{\rho}(\mathbf{R}, \mu, \theta) \times \mathbf{B}(\mathbf{R})). \quad (78)$$

The remaining terms in the Taylor expansions will be employed in subsequent subsections and are computed in the appendices.

We write the collision operator in the zeroth-order long-wavelength Fokker-Planck equation by employing the pull-back of (78) and its inverse, so the equation reads:

$$\begin{aligned} u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} F_{\sigma 0} - \hat{\mathbf{b}} \cdot (\mu \nabla_{\mathbf{R}} B + Z_\sigma \nabla_{\mathbf{R}} \varphi_0) \partial_u F_{\sigma 0} - B \partial_\theta F_{\sigma 1}^{\text{lw}} \\ = \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0}] (\mathbf{R}, u, \mu, \theta). \end{aligned} \quad (79)$$

Using that $F_{\sigma 0}$ is gyrophase independent and the isotropy property of the collision operator (by which it gives a gyrophase-independent function when acting on a gyrophase-independent function) we immediately deduce that

$$\partial_\theta F_{\sigma 1}^{\text{lw}} = 0, \quad (80)$$

i.e. $F_{\sigma 1}^{\text{lw}}$ is gyrophase-independent. Actually, it is trivial to prove from the zeroth-order short-wavelength component of equation (52) that also $\partial_\theta F_{\sigma 1}^{\text{sw}} = 0$, so

$$\partial_\theta F_{\sigma 1} = 0. \quad (81)$$

We proceed to prove that the solution to (79) is a stationary Maxwellian. Multiplying (79) by $-B \ln F_{\sigma 0}$ and integrating over u, μ , and θ :

$$\begin{aligned} -\nabla_{\mathbf{R}} \cdot \int \mathbf{B} u (F_{\sigma 0} \ln F_{\sigma 0} - F_{\sigma 0}) du d\mu d\theta \\ = - \int B \ln F_{\sigma 0} \sum_{\sigma'} \mathcal{T}_{0,\sigma}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0}] du d\mu d\theta. \end{aligned} \quad (82)$$

Here it is convenient to define the flux-surface average of a function $G(\psi, \Theta, \zeta)$, given by [40]

$$\langle G \rangle_\psi := \frac{\int_0^{2\pi} \int_0^{2\pi} \sqrt{g} G(\psi, \Theta, \zeta) d\Theta d\zeta}{\int_0^{2\pi} \int_0^{2\pi} \sqrt{g} d\Theta d\zeta}, \quad (83)$$

where

$$\sqrt{g} := \frac{1}{\nabla_{\mathbf{R}} \psi \cdot (\nabla_{\mathbf{R}} \Theta \times \nabla_{\mathbf{R}} \zeta)} \quad (84)$$

is the square root of the determinant of the metric tensor in coordinates $\{\psi, \Theta, \zeta\}$. It will also be useful to define the volume enclosed by the flux surface labeled by ψ ,

$$V(\psi) = \int_0^\psi d\psi \int_0^{2\pi} d\Theta \int_0^{2\pi} d\zeta \sqrt{g}. \quad (85)$$

The flux-surface average of (82) yields

$$\left\langle - \sum_{\sigma'} \int B \ln F_{\sigma 0} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0}] du d\mu d\theta \right\rangle_\psi = 0, \quad (86)$$

and after multiplying by τ_σ and adding over σ :

$$\left\langle - \sum_{\sigma, \sigma'} \tau_\sigma \int B \ln F_{\sigma 0} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0}] du d\mu d\theta \right\rangle_\psi = 0. \quad (87)$$

Observing that the Jacobian of $\mathcal{T}_{\sigma,0}$ at the point $(\mathbf{R}, u, \mu, \theta)$ is exactly $B(\mathbf{R})$, and using the formula for the change of variables in an integral, we obtain:

$$\left\langle - \sum_{\sigma, \sigma'} \tau_\sigma \int \ln (\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}) C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0}] d^3 v \right\rangle_\psi = 0. \quad (88)$$

This equation can be written as

$$\begin{aligned} & \left\langle \sum_{\sigma, \sigma'} \frac{\gamma_{\sigma\sigma'}}{2} \int \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \left(\tau_\sigma \nabla_{\mathbf{v}} \ln \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \right. \right. \\ & \left. \left. - \tau_{\sigma'} \nabla_{\mathbf{v}'} \ln \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right) \cdot \overset{\leftrightarrow}{\mathbf{W}} \cdot \left(\tau_\sigma \nabla_{\mathbf{v}} \ln \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \right. \right. \\ & \left. \left. - \tau_{\sigma'} \nabla_{\mathbf{v}'} \ln \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right) d^3 v d^3 v' \right\rangle_\psi = 0, \end{aligned} \quad (89)$$

which is the entropy production in a flux surface. The only solution to this equation is

$$\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}(\mathbf{r}, \mathbf{v}, t) = \frac{n_\sigma(\mathbf{r}, t)}{(2\pi T_\sigma(\mathbf{r}, t))^{3/2}} \exp \left(- \frac{(\mathbf{v} - \mathbf{V}(\mathbf{r}, t))^2}{2T_\sigma(\mathbf{r}, t)} \right), \quad (90)$$

where the temperature has to be the same for all the species (with the exception of electrons if a subsidiary expansion in the mass ratio is performed, or equivalently, if $\tau_e \sim \lambda_e \gg 1$ is used). That is, in the previous equation, $T_\sigma = T_{\sigma'}$ for every pair σ, σ' . Then,

$$\begin{aligned} F_{\sigma 0}(\mathbf{R}, u, \mu, t) = \\ \frac{n_\sigma(\mathbf{R}, t)}{(2\pi T_\sigma(\mathbf{R}, t))^{3/2}} \exp \left(- \frac{\mu B(\mathbf{R}) + (u - V_{||}(\mathbf{R}, t))^2/2}{T_\sigma(\mathbf{R}, t)} \right). \end{aligned} \quad (91)$$

In the last expression we have made explicit the fact that the component of $\mathbf{V}(\mathbf{R}, t)$ perpendicular to the magnetic field has to be zero because otherwise $F_{\sigma 0}$ would depend on the gyrophase; that is, $\mathbf{V}(\mathbf{R}, t) = V_{||}(\mathbf{R}, t) \hat{\mathbf{b}}(\mathbf{R}, t)$. Now, take the gyroaverage of (79)

and use (B.7) along with (91) to obtain

$$\begin{aligned}
u \hat{\mathbf{b}} \cdot & \left[\frac{1}{n_\sigma} \nabla_{\mathbf{R}} n_\sigma + \left(\frac{\mu B + (u - V_{\parallel})^2/2}{T_\sigma} - \frac{3}{2} \right) \frac{1}{T_\sigma} \nabla_{\mathbf{R}} T_\sigma \right. \\
& \left. + \frac{1}{T_\sigma} ((u - V_{\parallel}) \nabla_{\mathbf{R}} V_{\parallel} + Z_\sigma \nabla_{\mathbf{R}} \varphi_0) \right] \\
& - \frac{V_{\parallel}}{T_\sigma} \hat{\mathbf{b}} \cdot (\mu \nabla_{\mathbf{R}} B + Z_\sigma \nabla_{\mathbf{R}} \varphi_0) = 0.
\end{aligned} \tag{92}$$

Since this equation has to be satisfied for every u and μ , V_{\parallel} must vanish identically and T_σ must be a flux function. Then, from (92), we infer that the combination

$$\eta_\sigma = n_\sigma \exp \left(\frac{Z_\sigma \varphi_0}{T_\sigma} \right) \tag{93}$$

is a function of ψ and t only, $\eta_\sigma(\psi, t)$. The zeroth-order long-wavelength quasineutrality equation (see (122) later on in this paper) gives

$$\sum_{\sigma} Z_\sigma n_\sigma = 0, \tag{94}$$

or equivalently,

$$\sum_{\sigma} Z_\sigma \eta_\sigma \exp \left(-\frac{Z_\sigma \varphi_0}{T_\sigma} \right) = 0. \tag{95}$$

Taking the parallel gradient of this equation, one shows that φ_0 and n_σ are flux functions.

3.3. Long-wavelength Fokker-Planck equation to $O(\epsilon_\sigma)$

The equation to order ϵ_σ is fairly more complicated. Apart from the material in Appendix A, we need the long-wavelength limit of the pull-back of $F_{\sigma 0}$ by \mathcal{T}_σ^{-1} to order ϵ_σ . This is computed in Appendix C (see (C.9)). The result is

$$\begin{aligned}
& \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) F_{\sigma 1}^{\text{lw}} - B \partial_\theta F_{\sigma 2}^{\text{lw}} \\
& + \left(\mathbf{v}_\kappa + \mathbf{v}_{\nabla B} + \mathbf{v}_{E,\sigma}^{(0)} \right) \cdot \nabla_{\mathbf{R}} F_{\sigma 0} \\
& + \left[u \boldsymbol{\kappa} \cdot \left(\mathbf{v}_{\nabla B} + \mathbf{v}_{E,\sigma}^{(0)} \right) - Z_\sigma \lambda_\sigma \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \right] \partial_u F_{\sigma 0} \\
& = \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\frac{1}{T_\sigma} \left(\mathbf{v} \cdot \mathbf{V}_\sigma^p + \left(\frac{v^2}{2T_\sigma} - \frac{5}{2} \right) \mathbf{v} \cdot \mathbf{V}_\sigma^T \right) \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \right. \\
& \quad \left. + \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 1}^{\text{lw}}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] + \sum_{\sigma'} \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \frac{1}{T_{\sigma'}} \left(\mathbf{v} \cdot \mathbf{V}_{\sigma'}^p \right. \right. \\
& \quad \left. \left. + \left(\frac{v^2}{2T_{\sigma'}} - \frac{5}{2} \right) \mathbf{v} \cdot \mathbf{V}_{\sigma'}^T \right) \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} + \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 1}^{\text{lw}} \right],
\end{aligned} \tag{96}$$

where

$$\mathbf{v}_\kappa := \frac{u^2}{B} \hat{\mathbf{b}} \times \boldsymbol{\kappa}, \quad (97)$$

$$\mathbf{v}_{\nabla B} := \frac{\mu}{B} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B, \quad (98)$$

$$\mathbf{v}_{E,\sigma}^{(0)} := \frac{Z_\sigma}{B} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \varphi_0, \quad (99)$$

and

$$\boldsymbol{\kappa} := \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \quad (100)$$

is the magnetic field curvature. The velocities \mathbf{V}_σ^p and \mathbf{V}_σ^T are defined by

$$\mathbf{V}_\sigma^p := \frac{1}{n_\sigma B} \hat{\mathbf{b}} \times \nabla p_\sigma, \quad \mathbf{V}_\sigma^T := \frac{1}{B} \hat{\mathbf{b}} \times \nabla T_\sigma. \quad (101)$$

Here, $p_\sigma := n_\sigma T_\sigma$ is the pressure of species σ . On the right-hand side of (96) we have employed (B.9) to prove that the contribution of φ_0 appearing in (C.9) vanishes within the collision operator. It is easy to find the equation for the gyrophase-dependent piece of $F_{\sigma 2}^{\text{lw}}$:

$$\begin{aligned} & -B \partial_\theta (F_{\sigma 2}^{\text{lw}} - \langle F_{\sigma 2}^{\text{lw}} \rangle) \\ &= \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\frac{1}{T_\sigma} \left(\mathbf{v} \cdot \mathbf{V}_\sigma^p \right. \right. \\ & \quad \left. \left. + \left(\frac{v^2}{2T_\sigma} - \frac{5}{2} \right) \mathbf{v} \cdot \mathbf{V}_\sigma^T \right) \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right] \\ & \quad + \sum_{\sigma'} \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \frac{1}{T_{\sigma'}} \left(\mathbf{v} \cdot \mathbf{V}_{\sigma'}^p \right. \right. \\ & \quad \left. \left. + \left(\frac{v^2}{2T_{\sigma'}} - \frac{5}{2} \right) \mathbf{v} \cdot \mathbf{V}_{\sigma'}^T \right) \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right]. \end{aligned} \quad (102)$$

The gyroaverage of (96) yields an equation for $F_{\sigma 1}^{\text{lw}}$ (recall from Section 3.2 that $F_{\sigma 1}$ is gyrophase-independent):

$$\begin{aligned} & \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) F_{\sigma 1}^{\text{lw}} \\ & \quad + \left(\mathbf{v}_\kappa + \mathbf{v}_{\nabla B} + \mathbf{v}_{E,\sigma}^{(0)} \right) \cdot \nabla_{\mathbf{R}} F_{\sigma 0} \\ & \quad + \left[u \boldsymbol{\kappa} \cdot \left(\mathbf{v}_{\nabla B} + \mathbf{v}_{E,\sigma}^{(0)} \right) - Z_\sigma \lambda_\sigma \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \right] \partial_u F_{\sigma 0} \\ &= \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 1}^{\text{lw}}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right] \\ & \quad + \sum_{\sigma'} \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'1}^{\text{lw}} \right]. \end{aligned} \quad (103)$$

Up to this point our computations are valid for an arbitrary time-independent magnetic field with nested flux surfaces. Now, we particularize to the case of an equilibrium tokamak magnetic field:

$$\mathbf{B} = I(\psi)\nabla_{\mathbf{R}}\zeta + \nabla_{\mathbf{R}}\zeta \times \nabla_{\mathbf{R}}\psi. \quad (104)$$

In Appendix D we show that, in this setting, equation (103) can be written as

$$\begin{aligned} & \left(u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}B\partial_u \right) G_{\sigma 1}^{\text{lw}} \\ &= \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} \left(G_{\sigma 1}^{\text{lw}} - \frac{Iu}{B} \Upsilon_{\sigma} F_{\sigma 0} \right), \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \\ &+ \sum_{\sigma'} \frac{\lambda_{\sigma}}{\lambda_{\sigma'}} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{0,\sigma}^{-1*} F_{\sigma 0}, \mathcal{T}_{0,\sigma'}^{-1*} \left(G_{\sigma' 1}^{\text{lw}} - \frac{Iu}{B} \Upsilon_{\sigma'} F_{\sigma' 0} \right) \right], \end{aligned} \quad (105)$$

where

$$G_{\sigma 1}^{\text{lw}} := F_{\sigma 1}^{\text{lw}} + \left\{ \frac{Z_{\sigma}\lambda_{\sigma}}{T_{\sigma}} \varphi_1^{\text{lw}} + \frac{Iu}{B} \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} \right) \right\} F_{\sigma 0}, \quad (106)$$

and

$$\Upsilon_{\sigma} := \partial_{\psi} \ln n_{\sigma} + \left(\frac{u^2/2 + \mu B}{T_{\sigma}} - \frac{3}{2} \right) \partial_{\psi} \ln T_{\sigma}. \quad (107)$$

It is a remarkable fact that in terms of the functions $G_{\sigma 1}^{\text{lw}}$ the first-order Fokker-Planck equations do not involve the electrostatic potential. Equation (105) is in a form that makes it easy to compare with the results of neoclassical theory [26, 27].

3.4. Short-wavelength Fokker-Planck and quasineutrality equations to $O(\epsilon_{\sigma})$

Here, the equations for $F_{\sigma 1}^{\text{sw}}$ and $\phi_{\sigma 1}^{\text{sw}}$ are given because they enter the second-order, long-wavelength piece of the Fokker-Planck equation. Before presenting such short-wavelength equations, we need to define a new operator $\mathbb{T}_{\sigma,0}$ acting on phase-space functions $F(\mathbf{R}, u, \mu, \theta)$. Namely,

$$\mathbb{T}_{\sigma,0} F(\mathbf{r}, \mathbf{v}) := F \left(\mathbf{r} - \epsilon_{\sigma} \mathcal{T}_{\sigma,0}^{-1*} \boldsymbol{\rho}(\mathbf{r}, \mathbf{v}), \mathbf{v} \cdot \hat{\mathbf{b}}(\mathbf{r}), \frac{v_{\perp}^2}{2B(\mathbf{r})}, \arctan \left(\frac{\mathbf{v} \cdot \hat{\mathbf{e}}_2(\mathbf{r})}{\mathbf{v} \cdot \hat{\mathbf{e}}_1(\mathbf{r})} \right) \right). \quad (108)$$

This operator is useful to write some expressions involving the short wavelength pieces of the distribution function and the potential, for which it is not possible to Taylor expand the dependence on $\mathbf{r} - \epsilon_{\sigma} \mathcal{T}_{\sigma,0}^{-1*} \boldsymbol{\rho}(\mathbf{r}, \mathbf{v})$ around \mathbf{r} .

The first-order, short-wavelength terms of (52) yield

$$\begin{aligned} & \frac{1}{\tau_{\sigma}} \partial_t F_{\sigma 1}^{\text{sw}} + \left(u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}B\partial_u \right) F_{\sigma 1}^{\text{sw}} \\ &+ \left[\frac{Z_{\sigma}\lambda_{\sigma}}{B} \left(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_{\perp}/\epsilon_{\sigma}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right) \cdot \nabla_{\mathbf{R}_{\perp}/\epsilon_{\sigma}} F_{\sigma 1}^{\text{sw}} \right]^{\text{sw}} \\ &+ \left(\frac{u^2}{B} \hat{\mathbf{b}} \times \boldsymbol{\kappa} + \frac{\mu}{B} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}}B + \frac{Z_{\sigma}}{B} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}}\varphi_0 \right) \cdot \nabla_{\mathbf{R}_{\perp}/\epsilon_{\sigma}} F_{\sigma 1}^{\text{sw}} \end{aligned}$$

$$\begin{aligned}
& + \frac{Z_\sigma \lambda_\sigma}{B} \left(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right) \cdot \nabla_{\mathbf{R}} F_{\sigma 0} \\
& - Z_\sigma \lambda_\sigma \left(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle + \frac{u}{B} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right) \partial_u F_{\sigma 0} \\
& = \sum_{\sigma'} \left\langle \mathcal{T}_{NP, \sigma}^* C_{\sigma \sigma'} \left[\mathbb{T}_{\sigma, 0} F_{\sigma 1}^{\text{sw}} - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma, 0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma, 0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma', 0}^{-1*} F_{\sigma' 0} \right] \right\rangle \\
& + \sum_{\sigma'} \frac{\lambda_\sigma}{\lambda_{\sigma'}} \left\langle \mathcal{T}_{NP, \sigma}^* C_{\sigma \sigma'} \left[\mathcal{T}_{\sigma, 0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma', 0} F_{\sigma' 1}^{\text{sw}} - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma', 0} \tilde{\phi}_{\sigma' 1}^{\text{sw}} \mathcal{T}_{\sigma', 0}^{-1*} F_{\sigma' 0} \right] \right\rangle. \tag{109}
\end{aligned}$$

As for the short-wavelength, first-order quasineutrality equation:

$$\begin{aligned}
& \sum_{\sigma} \frac{Z_\sigma}{\lambda_\sigma} \int B \left[-Z_\sigma \lambda_\sigma \tilde{\phi}_{\sigma 1}^{\text{sw}} (\mathbf{r} - \epsilon_\sigma \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), \mu, \theta, t) \frac{F_{\sigma 0}(\mathbf{r}, u, \mu, t)}{T_\sigma(\mathbf{r}, t)} \right. \\
& \left. + F_{\sigma 1}^{\text{sw}} (\mathbf{r} - \epsilon_\sigma \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), u, \mu, t) \right] du d\mu d\theta = 0. \tag{110}
\end{aligned}$$

3.5. Long-wavelength Fokker-Planck equation to $O(\epsilon_\sigma^2)$

The second-order contribution to (52) is cumbersome. In order to avoid lengthy calculations to those readers interested in reaching quickly the final expressions and main results, most of the manipulations in this subsection are deferred to the appendices.

The pieces of order ϵ_σ^2 in (52) yield

$$\begin{aligned}
& \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) F_{\sigma 2}^{\text{lw}} - B \partial_\theta F_{\sigma 3}^{\text{lw}} + \frac{\lambda_\sigma^2}{\tau_\sigma} \partial_{\epsilon_s^2 t} F_{\sigma 0} \\
& + \left(\mathbf{v}_\kappa + \mathbf{v}_{\nabla B} + \mathbf{v}_{E, \sigma}^{(0)} \right) \cdot \nabla_{\mathbf{R}} F_{\sigma 1}^{\text{lw}} \\
& + \left[-Z_\sigma \lambda_\sigma \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} + u \boldsymbol{\kappa} \cdot \left(\mathbf{v}_{\nabla B} + \mathbf{v}_{E, \sigma}^{(0)} \right) \right] \partial_u F_{\sigma 1}^{\text{lw}} \\
& + \left[\mathbf{v}_{E, \sigma}^{(1)} - \frac{u}{B} (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \left(\mathbf{v}_\kappa + \mathbf{v}_{\nabla B} + \mathbf{v}_{E, \sigma}^{(0)} \right) \right. \\
& \left. - \frac{u \mu}{B} (\nabla_{\mathbf{R}} \times \mathbf{K})_\perp + Z_\sigma \lambda_\sigma \partial_u \Psi_{\phi B, \sigma}^{\text{lw}} \hat{\mathbf{b}} + \partial_u \Psi_{B, \sigma} \hat{\mathbf{b}} \right] \cdot \nabla_{\mathbf{R}} F_{\sigma 0} \\
& - \left\{ Z_\sigma \lambda_\sigma^2 \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \left[\varphi_2^{\text{lw}} + \frac{\mu}{2 \lambda_\sigma^2 B} (\vec{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \varphi_0 \right] \right. \\
& + \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Psi_{B, \sigma} + Z_\sigma \lambda_\sigma \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Psi_{\phi B}^{\text{lw}} + Z_\sigma^2 \lambda_\sigma^2 \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Psi_{\phi}^{\text{lw}} \\
& \left. - u \boldsymbol{\kappa} \cdot \mathbf{v}_{E, \sigma}^{(1)} + \left[\frac{u^2}{B} (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \boldsymbol{\kappa} \right. \right. \\
& \left. \left. + \mu \left((\nabla_{\mathbf{R}} \times \mathbf{K}) \times \hat{\mathbf{b}} \right) \right] \cdot \left(\mathbf{v}_{\nabla B} + \mathbf{v}_{E, \sigma}^{(0)} \right) \right\} \partial_u F_{\sigma 0} \\
& + \frac{Z_\sigma \lambda_\sigma}{B} \left[\nabla_{\mathbf{R}} \cdot \left(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle F_{\sigma 1}^{\text{sw}} \right) \right]^{\text{lw}}
\end{aligned}$$

$$\begin{aligned}
& -Z_\sigma \lambda_\sigma \partial_u \left[\left(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle + \frac{u}{B} \left(\hat{\mathbf{b}} \times \boldsymbol{\kappa} \right) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right) F_{\sigma 1}^{\text{sw}} \right]^{\text{lw}} \\
& = \sum_{\sigma'} \left[\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{(1)} \right]^{\text{lw}} + \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(2)\text{lw}}.
\end{aligned} \tag{111}$$

Here,

$$\mathbf{v}_{E,\sigma}^{(1)} = \frac{Z_\sigma \lambda_\sigma}{B} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp} \varphi_1^{\text{lw}}, \tag{112}$$

$$\Psi_{\phi B}^{\text{lw}} = -\frac{3\mu}{2\lambda_\sigma B^2} \nabla_{\mathbf{R}} B \cdot \nabla_{\mathbf{R}} \varphi_0 - \frac{u^2}{\lambda_\sigma B^2} (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \varphi_0, \tag{113}$$

and

$$\Psi_\phi^{\text{lw}} = -\frac{1}{2\lambda_\sigma^2 B^2} |\nabla_{\mathbf{R}} \varphi_0|^2 - \frac{1}{2B} \partial_\mu \left[\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \rangle \right]^{\text{lw}}. \tag{114}$$

As for the collision operator,

$$C_{\sigma\sigma'}^{(1)} = C_{\sigma\sigma'}^{(1)\text{lw}} + C_{\sigma\sigma'}^{(1)\text{sw}}, \tag{115}$$

$$\begin{aligned}
\left[\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{(1)} \right]^{\text{lw}} & = \left(\boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} + \hat{u}_1 \partial_u + \hat{\mu}_1^{\text{lw}} \partial_\mu + \hat{\theta}_1^{\text{lw}} \partial_\theta \right) \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(1)\text{lw}} \\
& + \left[\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{(1)\text{sw}} \right]^{\text{lw}},
\end{aligned} \tag{116}$$

where

$$\begin{aligned}
C_{\sigma\sigma'}^{(1)\text{lw}} & = C_{\sigma\sigma'} \left[\frac{1}{T_\sigma} \mathbf{v} \cdot \left(\mathbf{V}_\sigma^p + \left(\frac{v^2}{2T_\sigma} - \frac{5}{2} \right) \mathbf{V}_\sigma^T \right) \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \right. \\
& + \mathcal{T}_{\sigma,0}^{-1*} \left(G_{\sigma 1} - \frac{Iu}{B} \Upsilon_\sigma F_{\sigma 0} \right), \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \left. \right] \\
& + \frac{\lambda_\sigma}{\lambda_{\sigma'}} C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \frac{1}{T_{\sigma'}} \mathbf{v} \cdot \left(\mathbf{V}_{\sigma'}^p + \left(\frac{v^2}{2T_{\sigma'}} - \frac{5}{2} \right) \mathbf{V}_{\sigma'}^T \right) \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right. \\
& + \mathcal{T}_{\sigma',0}^{-1*} \left(G_{\sigma'1} - \frac{Iu}{B} \Upsilon_{\sigma'} F_{\sigma'0} \right) \left. \right],
\end{aligned} \tag{117}$$

and

$$\begin{aligned}
C_{\sigma\sigma'}^{(1)\text{sw}} & = C_{\sigma\sigma'} \left[\mathbb{T}_{\sigma,0} F_{\sigma 1}^{\text{sw}} - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma,0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right] \\
& + \frac{\lambda_\sigma}{\lambda_{\sigma'}} C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma',0} F_{\sigma'1}^{\text{sw}} - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma',0} \tilde{\phi}_{\sigma'1}^{\text{sw}} \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right].
\end{aligned} \tag{118}$$

The left-hand side of (111) is written by employing again the Poisson brackets obtained in Appendix A. The first-order coordinate transformation that enters explicitly the expression of the collision operator is computed in detail in Appendix C; in particular, $\hat{u}_1, \hat{\mu}_1^{\text{lw}}, \hat{\theta}_1^{\text{lw}}$ are defined in (C.7) and (C.8). In Appendix E we calculate the last term of (116), $[\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{(1)\text{sw}}]^{\text{lw}}$, and its gyroaverage. Finally, $C_{\sigma\sigma'}^{(2)\text{lw}}$ is calculated in Appendix F.

In Appendix H we prove that the gyroaverage of (111) can be rearranged so that it reads

$$\begin{aligned}
& \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma 2}^{\text{lw}} + \frac{\lambda_\sigma^2}{\tau_\sigma} \partial_{\epsilon_s^2 t} F_{\sigma 0} \\
& - \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta \partial_\psi \left\{ \frac{Z_\sigma \lambda_\sigma}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \left[F_{\sigma 1}^{\text{sw}} (\nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \psi \right]^{\text{lw}} \right. \\
& + \frac{1}{\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta} \left\langle \left(\frac{Iu}{B} + \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \psi \right) \sum_{\sigma'} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \Bigg\} \\
& - \partial_u \left\{ \left[Z_\sigma \lambda_\sigma F_{\sigma 1}^{\text{sw}} (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right. \right. \\
& + \frac{\mu}{uB} \partial_\Theta B (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \Theta) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \\
& + \frac{u}{B} [\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \Bigg] \Bigg\}^{\text{lw}} \\
& - \left\langle \frac{I}{B} (\mu \partial_\psi B + Z_\sigma \partial_\psi \varphi_0) \sum_{\sigma'} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \Bigg\} \\
& + \partial_\mu \left\langle \frac{1}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \psi (\mu \partial_\psi B + Z_\sigma \partial_\psi \varphi_0) \sum_{\sigma'} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \\
& = - \sum_{\sigma'} \partial_u \left\langle \left[\frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{u} + \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right. \right. \\
& - \frac{1}{u} \boldsymbol{\rho} \cdot (\mu \partial_\Theta B \nabla_{\mathbf{R}} \Theta + u^2 \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \Bigg] \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \Bigg\rangle \\
& + \sum_{\sigma'} \partial_\mu \left\langle \left[\frac{u\mu}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right. \right. \\
& - \frac{1}{B} \boldsymbol{\rho} \cdot (\mu \partial_\Theta B \nabla_{\mathbf{R}} \Theta + u^2 \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \Bigg] \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \Bigg\rangle \\
& + \sum_{\sigma'} \left[\left\langle \mathcal{T}_{\sigma, 1}^* C_{\sigma \sigma'}^{(1)\text{sw}} \right\rangle \right]^{\text{lw}} + \sum_{\sigma'} \left\langle \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(2)\text{lw}} \right\rangle, \tag{119}
\end{aligned}$$

where $G_{\sigma 2}^{\text{lw}}$ is defined in (H.25). Note that the first-order, short-wavelength pieces of the distribution function and electrostatic potential, $F_{\sigma 1}^{\text{sw}}$ and $\phi_{\sigma 1}^{\text{sw}}$, enter equation (119). The equations needed to determine them are given in subsection 3.4. Observe also that the time derivative of $F_{\sigma 0}$ appears in (119), something that has very important consequences. We will learn that equation (119) has non-trivial solvability conditions that involve the time evolution of certain moments of $F_{\sigma 0}$.

The function $G_{\sigma 2}^{\text{lw}}$ is defined to make comparisons with neoclassical theory easier. It is also useful to obtain the solvability conditions in subsection 5.2 with less algebra.

4. Long-wavelength quasineutrality equation

In this section we obtain the quasineutrality equation, (65), at long-wavelengths. For convenience, we repeat here equation (65):

$$\sum_{\sigma} Z_{\sigma} \int B_{\parallel, \sigma}^* F_{\sigma} \delta\left(\pi^{\mathbf{r}}\left(\mathcal{T}_{\sigma}(\mathbf{R}, u, \mu, \theta, t)\right) - \mathbf{r}\right) d^3 R du d\mu d\theta = 0, \quad (120)$$

with $\pi^{\mathbf{r}}(\mathbf{r}, \mathbf{v}) := \mathbf{r}$.

At long wavelengths we can simply expand the argument of the Dirac delta function around $\mathbf{R} - \mathbf{r}$. Using that

$$\pi^{\mathbf{r}} \mathcal{T}_{\sigma}(\mathbf{R}, u, \mu, \theta, t) = \mathbf{R} + \epsilon_{\sigma} \boldsymbol{\rho} + \epsilon_{\sigma}^2 (\mathbf{R}_{\sigma, 2} + \mu_{\sigma, 1} \partial_{\mu} \boldsymbol{\rho} + \theta_{\sigma, 1} \partial_{\theta} \boldsymbol{\rho}) + O(\epsilon_{\sigma}^3), \quad (121)$$

that the first-order term of $B_{\parallel, \sigma}^*$ is odd in u , that $F_{\sigma 0}$ is even in u , that $F_{\sigma 1}^{\text{lw}}$ does not depend on θ , and integrating over \mathbf{R} , it is straightforward to obtain

$$\sum_{\sigma} Z_{\sigma} n_{\sigma}(\mathbf{r}, t) = 0 \quad (122)$$

to order ϵ_s^0 ,

$$\sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}} \int B(\mathbf{r}) F_{\sigma 1}^{\text{lw}}(\mathbf{r}, u, \mu, t) du d\mu d\theta = 0 \quad (123)$$

to order ϵ_s , and

$$\begin{aligned} \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}^2} \left[\int (B F_{\sigma 2}^{\text{lw}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \mathbf{K} F_{\sigma 0} + u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \hat{\mathbf{b}} F_{\sigma 1}^{\text{lw}}) du d\mu d\theta \right. \\ \left. - \nabla_{\mathbf{r}} \cdot \int (\mathbf{R}_{\sigma, 2}^{\text{lw}} + \mu_{\sigma, 1}^{\text{lw}} \partial_{\mu} \boldsymbol{\rho} + \theta_{\sigma, 1}^{\text{lw}} \partial_{\theta} \boldsymbol{\rho}) B F_{\sigma 0} du d\mu d\theta \right. \\ \left. + \frac{1}{2} \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} : \int \boldsymbol{\rho} \boldsymbol{\rho} B F_{\sigma 0} du d\mu d\theta \right] \end{aligned} \quad (124)$$

to order ϵ_s^2 . Here everything is evaluated at $\mathbf{R} = \mathbf{r}$. In writing the arguments of some functions we have stressed that they are evaluated at $\mathbf{R} = \mathbf{r}$, e.g. $n_{\sigma}(\mathbf{r})$, but we should not forget that n_{σ} , for example, depends only on ψ in flux coordinates. Note that to be formally correct we need a unique, species-independent expansion parameter, and we have chosen ϵ_s as indicated in Section 2. In Appendix J we show that (124) can be transformed into

$$\begin{aligned} \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}^2} \left[\int (B F_{\sigma 2}^{\text{lw}} + u \hat{\mathbf{b}} \cdot (\nabla_{\mathbf{r}} \times \hat{\mathbf{b}}) F_{\sigma 1}^{\text{lw}}) du d\mu d\theta \right. \\ \left. - \hat{\mathbf{b}} \cdot (\nabla_{\mathbf{r}} \times \mathbf{K}) \frac{n_{\sigma} T_{\sigma}}{B^2} + \nabla_{\mathbf{r}} \cdot \left(\frac{3}{2} \frac{\nabla_{\mathbf{r} \perp} B}{B^3} n_{\sigma} T_{\sigma} \right) \right. \\ \left. + \frac{1}{2} \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} : \left(\left(\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}} \right) \frac{n_{\sigma} T_{\sigma}}{B^2} \right) \right. \\ \left. + \nabla_{\mathbf{r}} \cdot \left(\left(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \right) \frac{n_{\sigma} T_{\sigma}}{B^2} \right) + \nabla_{\mathbf{r}} \cdot \left(\frac{Z_{\sigma} n_{\sigma}}{B^2} \nabla_{\mathbf{r}} \varphi_0 \right) \right] = 0. \end{aligned} \quad (125)$$

Observe that the above expressions for the long-wavelength quasineutrality equation are completely general, i.e. we have not particularized for tokamak geometry. We proceed to do it next by writing (123) and (125) in terms of the functions $G_{\sigma 1}^{\text{lw}}$ and $G_{\sigma 2}^{\text{lw}}$, defined in (106) and (H.25). This is obvious for the first-order piece of the quasineutrality equation, yielding

$$\sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}} \left(\int B(\mathbf{r}) G_{\sigma 1}^{\text{lw}}(\mathbf{r}, u, \mu, t) du d\mu d\theta - \frac{Z_{\sigma} \lambda_{\sigma}}{T_{\sigma}} n_{\sigma}(\mathbf{r}, t) \varphi_1^{\text{lw}}(\mathbf{r}, t) \right) = 0, \quad (126)$$

and in Appendix J it is shown that the result for the second-order piece is

$$\begin{aligned} \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}^2} \int B \Big[& G_{\sigma 2}^{\text{lw}} + \frac{u}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \hat{\mathbf{b}} G_{\sigma 1}^{\text{lw}} \\ & + \left(\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{u} \partial_u - \frac{I u}{B} \partial_{\psi} \right) G_{\sigma 1}^{\text{lw}} \\ & - \left[\frac{Z_{\sigma} \lambda_{\sigma}}{u \mathbf{B} \cdot \nabla_{\mathbf{r}} \Theta} F_{\sigma 1}^{\text{sw}} (\hat{\mathbf{b}} \times \nabla_{\mathbf{r} \perp / \epsilon_{\sigma}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle) \cdot \nabla_{\mathbf{r}} \Theta \right]^{\text{lw}} \\ & + \left\langle \frac{1}{u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \Theta} \boldsymbol{\rho} \cdot \nabla_{\mathbf{r}} \Theta \sum_{\sigma'} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1) \text{lw}} \right\rangle \\ & + \frac{Z_{\sigma}^2 \lambda_{\sigma}^2}{2 T_{\sigma}^2} \left[\left\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right\rangle \right]^{\text{lw}} F_{\sigma 0} \Big] du d\mu d\theta \\ & + \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}^2} n_{\sigma} T_{\sigma} \left\{ \frac{Z_{\sigma} \lambda_{\sigma}^2}{T_{\sigma}^2} \left(\frac{Z_{\sigma}}{2 T_{\sigma}} (\varphi_1^{\text{lw}})^2 - \varphi_2^{\text{lw}} \right) \right. \\ & + \frac{R^2}{2} \left[\left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \partial_{\psi} \ln n_{\sigma} \right)^2 \right. \\ & + (\partial_{\psi} \ln T_{\sigma})^2 + 2 \partial_{\psi} \ln n_{\sigma} \partial_{\psi} \ln T_{\sigma} \\ & \left. \left. + \partial_{\psi}^2 \ln n_{\sigma} + \frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi}^2 \varphi_0 + \partial_{\psi}^2 \ln T_{\sigma} \right] \right\} = 0, \quad (127) \end{aligned}$$

where R is the major radius coordinate, i.e. it is the distance to the axis of symmetry of the tokamak.

5. Indeterminacy of the long-wavelength radial electric field

With the results of Sections 3 and 4 at hand it is reasonably easy to prove that in a tokamak $\varphi_0(\psi)$ is not determined by second-order Fokker-Planck and quasineutrality equations. In order to be as clear as possible, we divide the argument into three steps. In subsection 5.1 we show that the quasineutrality equation gives no information about the radial electric field, even though naively one would have expected to use this equation to solve for it. In subsection 5.2 we learn that (119) possesses non-trivial solvability

conditions and work them out. They are transport equations for the lowest order density and temperature functions. In subsection 5.3 we prove that these solvability conditions do not yield new equations for the radial electric field. The proof amounts to explicitly showing that the turbulent tokamak is intrinsically ambipolar.

5.1. Quasineutrality equation and long-wavelength radial electric field

It is obvious from equations (105) and (119) that if $G_{\sigma j}^{\text{lw}}$, $j = 1, 2$ are solutions of the first and second-order Fokker-Planck equations, then so are $G_{\sigma j}^{\text{lw}} + h_{\sigma j}$, $j = 1, 2$, where

$$h_{\sigma j} = \left[\frac{n_{\sigma j}}{n_\sigma} + \left(\frac{\mu B + u^2/2}{T_\sigma} - \frac{3}{2} \right) \frac{T_{\sigma j}}{T_\sigma} \right] F_{\sigma 0}, \quad (128)$$

for an arbitrary set of flux functions $\{n_{\sigma j}(\psi, t), T_{\sigma j}(\psi, t)\}_\sigma$, with the only restriction $T_{\sigma j}/\lambda_\sigma^j = T_{\sigma' j}/\lambda_{\sigma'}^j$, for all σ, σ' (the temperature of the electrons is allowed to be different if we expand in the mass ratio, that is, if we use $\lambda_e \sim \tau_e \gg 1$). In other words, the operator acting on $G_{\sigma 1}^{\text{lw}}$ in (105) and on $G_{\sigma 2}^{\text{lw}}$ in (119) has a kernel given by (128) with an obvious interpretation: it consists of corrections of order ϵ_σ^j to the zeroth-order particle densities, n_σ , and temperatures, T_σ . Therefore, in order to have a unique solution for the Fokker-Planck equation, one needs to prescribe a condition that eliminates the freedom introduced by the existence of a non-zero kernel. An example of such a condition is given by imposing, for $j = 1, 2$,

$$\begin{aligned} \int B G_{\sigma j} d\mathbf{u} d\mu d\theta &= 0, \text{ for every } \sigma, \text{ and} \\ \sum_\sigma \frac{1}{\lambda_\sigma^j} \int B (u^2/2 + \mu B) G_{\sigma j} d\mathbf{u} d\mu d\theta &= 0. \end{aligned} \quad (129)$$

Of course, even though this is a natural choice, there are infinitely many different possibilities.

Assume that $\mathbf{G}_{\sigma j}$, $j = 1, 2$ (note the different font) are particular solutions of (105) and (119) (not necessarily satisfying (129)). Then, any solution of (105) and (119) is of the form $\mathbf{G}_{\sigma j}^{\text{lw}} + h_{\sigma j}$, $j = 1, 2$. When introduced in (126) and (127) we find:

$$\begin{aligned} \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma} n_{\sigma 1} + \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma} \left(\int B(\mathbf{r}) \mathbf{G}_{\sigma 1}^{\text{lw}}(\mathbf{r}, u, \mu, \theta, t) d\mathbf{u} d\mu d\theta \right. \\ \left. - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} n_\sigma(\mathbf{r}) \varphi_1^{\text{lw}}(\mathbf{r}, t) \right) = 0, \end{aligned} \quad (130)$$

$$\begin{aligned} \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma^2} n_{\sigma 2} + \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma^2} \int B \left[\mathbf{G}_{\sigma 2}^{\text{lw}} + \frac{u}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \hat{\mathbf{b}} \mathbf{G}_{\sigma 1}^{\text{lw}} \right. \\ \left. + \left(\frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{u} \partial_u - \frac{Iu}{B} \partial_\psi \right) \mathbf{G}_{\sigma 1}^{\text{lw}} \right. \\ \left. - \left[\frac{Z_\sigma \lambda_\sigma}{u \mathbf{B} \cdot \nabla_{\mathbf{r}} \Theta} F_{\sigma 1}^{\text{sw}} (\hat{\mathbf{b}} \times \nabla_{\mathbf{r}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle) \cdot \nabla_{\mathbf{r}} \Theta \right] \right]^{\text{lw}} \end{aligned}$$

$$\begin{aligned}
& + \left\langle \frac{1}{u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \Theta} \boldsymbol{\rho} \cdot \nabla_{\mathbf{r}} \Theta \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(1)\text{lw}} \right\rangle \\
& + \frac{Z_{\sigma}^2 \lambda_{\sigma}^2}{2T_{\sigma}^2} \left[\left\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right\rangle \right]^{\text{lw}} F_{\sigma 0} \Big] du d\mu d\theta \\
& + \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}^2} n_{\sigma} T_{\sigma} \left\{ \frac{Z_{\sigma} \lambda_{\sigma}^2}{T_{\sigma}^2} \left(\frac{Z_{\sigma}}{2T_{\sigma}} (\varphi_1^{\text{lw}})^2 - \varphi_2^{\text{lw}} \right) \right. \\
& + \frac{R^2}{2} \left[\left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \partial_{\psi} \ln n_{\sigma} \right)^2 \right. \\
& + (\partial_{\psi} \ln T_{\sigma})^2 + 2 \partial_{\psi} \ln n_{\sigma} \partial_{\psi} \ln T_{\sigma} \\
& \left. \left. + \partial_{\psi}^2 \ln n_{\sigma} + \frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi}^2 \varphi_0 + \partial_{\psi}^2 \ln T_{\sigma} \right] \right\} = 0. \tag{131}
\end{aligned}$$

Even though φ_0 enters this equation to second-order, it cannot be determined. The first and second-order pieces of the long-wavelength quasineutrality equation simply give constraints on the corrections $n_{\sigma 1}$ and $n_{\sigma 2}$. Each function $n_{\sigma j}$ will be determined by a transport equation that appears as a solvability condition for a higher order long-wavelength piece of the Fokker-Planck equation, just as a transport equation for n_{σ} is derived in subsection 5.2.1 as a solvability condition for equation (119). Note that we cannot choose the value of $n_{\sigma j}$; we can only decide which piece of $G_{\sigma j}$ we call $n_{\sigma j}$ and which piece we leave within $\mathbf{G}_{\sigma j}$. The density corrections $n_{\sigma j}$ cannot be set to zero arbitrarily because we need them to satisfy the solvability conditions of the higher order pieces of the Fokker-Planck equation.||

We cannot calculate φ_0 from the quasineutrality equation to this order, but the first and second-order pieces of the long-wavelength poloidal electric field can be found, respectively, from (130) and (131). This can be viewed by acting on the latter equations with $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}}$ and employing that $n_{\sigma 1}$ and $n_{\sigma 2}$ are only functions of ψ . Not surprisingly, φ_1^{lw} and φ_2^{lw} are determined up to an arbitrary, additive function of ψ , that can be absorbed by redefining the corrections $n_{\sigma 1}$ and $n_{\sigma 2}$. Without loss of generality, we fix the ambiguity by taking

$$\langle \varphi_1^{\text{lw}} \rangle_{\psi} = 0 \tag{132}$$

and

$$\langle \varphi_2^{\text{lw}} \rangle_{\psi} = 0. \tag{133}$$

|| Our procedure here has diverged from the canonical Chapman-Enskog approach, where the density of the lowest order Maxwellian is not broken into pieces of different orders. Instead, in the Chapman-Enskog theory, the conservation equation for particle density, obtained from the solvability conditions (see subsection 5.2), contains terms of different orders. Our procedure is different in that the conservation equation will be split into different orders, each giving an equation for a piece $n_{\sigma j}$. In this way we highlight that the quasineutrality equation to $O(\epsilon_s^2)$ does not allow to solve for φ_0 because this would require knowing $n_{\sigma 1}$ and $n_{\sigma 2}$, which are determined from higher order pieces of the long-wavelength Fokker-Planck equation.

In Subsection 5.2 we explain that the second-order Fokker-Planck equation possesses some solvability conditions. We have to show that their fulfillment does not impose any additional conditions that give $\varphi_0(\psi, t)$, and we do so in subsection 5.3.

5.2. Transport equations

Some of the benefits of writing the Fokker-Planck equation precisely in the form (119) will be appreciated in this subsection, where we show that time evolution equations for the lowest order density and temperature functions n_σ and T_σ are obtained as solvability conditions for the second-order, long-wavelength Fokker-Planck equation. That is, we prove that if a solution for $G_{\sigma 2}^{\text{lw}}$ (equivalently, for $F_{\sigma 2}^{\text{lw}}$) exists, then (119) imposes certain constraints among lower-order quantities (solvability conditions). These conditions turn out to be transport equations for density and energy. In Appendix M we prove that these transport equations are indeed the only solvability conditions obtained from the Fokker-Planck equation up to order ϵ_s^2 .

5.2.1. Transport equation for density. Go back to (119), multiply by $\tau_\sigma B/\lambda_\sigma^2$, integrate over u, μ, θ and take the flux-surface average:

$$\begin{aligned} \partial_{\epsilon_s^2 t} n_\sigma(\psi, t) = & \frac{1}{V'(\psi)} \partial_\psi \left\langle V'(\psi) \int du d\mu d\theta \left\{ \right. \right. \\ & \left. \left[F_{\sigma 1}^{\text{sw}} (\nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \psi \right]^{\text{lw}} \right. \\ & \left. + \frac{B}{Z_\sigma \lambda_\sigma} \left\langle \left(\frac{Iu}{B} + \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \psi \right) \sum_{\sigma'} C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \right\} \right\rangle_\psi, \end{aligned} \quad (134)$$

where $V'(\psi)$ is the derivative of the function $V(\psi)$, defined in (85), which gives the volume enclosed by the flux surface with label ψ . We have also used that for the tokamak the square root of the determinant of the metric tensor (recall (84)) is

$$\sqrt{g} = \frac{1}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta}. \quad (135)$$

Equation (134) is a transport equation for each lowest-order particle density function n_σ . Note that φ_0 and φ_1^{lw} do not appear.

5.2.2. Transport equation for energy. Now, we do something similar for the total energy. Multiply (119) by $(\tau_\sigma/\lambda_\sigma^2)B(u^2/2 + \mu B)$, integrate over u, μ, θ , and take the flux-surface average. Then,

$$\begin{aligned} \partial_{\epsilon_s^2 t} \left(\frac{3}{2} n_\sigma(\psi, t) T_\sigma(\psi, t) \right) = & \\ & \frac{1}{V'(\psi)} \partial_\psi \left\langle V'(\psi) \int (u^2/2 + \mu B) \left\{ \right. \right. \\ & \left. \left[F_{\sigma 1}^{\text{sw}} (\nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \psi \right]^{\text{lw}} \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{B}{Z_\sigma \lambda_\sigma} \left\langle \left(\frac{Iu}{B} + \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \psi \right) \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(1)\text{lw}} \right\rangle \Bigg\rangle_\psi \\
& - \left\langle \int B \left[F_{\sigma 1}^{\text{sw}} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right. \right. \right. \\
& + \frac{\mu}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \\
& + \frac{u^2}{B} [\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \Bigg] \Bigg\rangle_\psi^{\text{lw}} \\
& + \frac{1}{\lambda_\sigma} \partial_\psi \varphi_0 \left\langle \int B \left(\frac{Iu}{B} + \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \psi \right) \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(1)\text{lw}} \right\rangle_\psi \\
& + \left\langle \frac{\tau_\sigma}{\lambda_\sigma^2} \int B (u^2/2 + \mu B) \sum_{\sigma'} \left[\left\langle \mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{(1)\text{sw}} \right\rangle \right]^{\text{lw}} \right. \\
& \left. + \left\langle \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(2)\text{lw}} \right\rangle \right] \text{d}u \text{d}\mu \text{d}\theta \Bigg\rangle_\psi, \tag{136}
\end{aligned}$$

which is a transport equation for the energy density of species σ . The term containing φ_1^{lw} in (119) does not contribute to (136) because the collision operator conserves the total number of particles of each species. Equation (136) gives an equation for the temperature of each species. Unless we expand in the mass ratio $\lambda_e \sim \tau_e \gg 1$, and allow different temperatures for electrons and ions, this equation still contains the function $G_{\sigma 2}^{\text{lw}}$ in $C_{\sigma\sigma'}^{(2)\text{lw}}$ and cannot be considered a solvability condition. It is possible to prove that for $\lambda_e \sim \tau_e \gg 1$, the equations for the electron and ion temperatures do not contain $G_{\sigma 2}^{\text{lw}}$ and consequently are independent solvability conditions that determine T_i and T_e . However, in general, the only way to eliminate $G_{\sigma 2}^{\text{lw}}$ is summing over all species. We obtain

$$\begin{aligned}
& \partial_{\epsilon_s^2 t} \left(\sum_\sigma \frac{3}{2} n_\sigma(\psi, t) T_\sigma(\psi, t) \right) = \\
& \frac{1}{V'(\psi)} \partial_\psi \left\langle V'(\psi) \int (u^2/2 + \mu B) \sum_\sigma \left\{ \right. \right. \\
& \left[F_{\sigma 1}^{\text{sw}} (\nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \psi \right]^{\text{lw}} \\
& + \frac{B}{Z_\sigma \lambda_\sigma} \left\langle \left(\frac{Iu}{B} + \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \psi \right) \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(1)\text{lw}} \right\rangle \Bigg\} \text{d}u \text{d}\mu \text{d}\theta \Bigg\rangle_\psi \\
& - \left\langle \sum_\sigma \int B \left[F_{\sigma 1}^{\text{sw}} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right. \right. \right. \\
& + \frac{\mu}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \\
& + \frac{u^2}{B} [\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \Bigg] \Bigg\rangle_\psi^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta \Bigg\rangle_\psi
\end{aligned}$$

$$+ \left\langle \sum_{\sigma, \sigma'} \frac{\tau_\sigma}{\lambda_\sigma^2} \int B (u^2/2 + \mu B) \left[\left\langle \mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{(1)\text{sw}} \right\rangle \right]^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta \right\rangle_\psi. \quad (137)$$

Here we have used the conservation of momentum and energy by the collision operator to find

$$\begin{aligned} \sum_{\sigma, \sigma'} \left\langle \frac{1}{\lambda_\sigma} \int B \left(\frac{Iu}{B} + \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \psi \right) \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(1)\text{lw}} \text{d}u \text{d}\mu \text{d}\theta \right\rangle_\psi &= 0, \\ \sum_{\sigma, \sigma'} \left\langle \frac{\tau_\sigma}{\lambda_\sigma^2} \int B (u^2/2 + \mu B) \left\langle \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(2)\text{lw}} \right\rangle \text{d}u \text{d}\mu \text{d}\theta \right\rangle_\psi &= 0. \end{aligned} \quad (138)$$

These expressions are easily deduced from the conservation properties of the collision operator (B.6) by realizing that the summations of collision operators in (138) can always be decomposed in binomials of the form

$$\begin{aligned} \int B \left(\frac{Iu}{B} + \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \psi \right) (\mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\lambda_\sigma^{-1} g_\sigma, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0}] \\ + \mathcal{T}_{\sigma',0}^* C_{\sigma'\sigma} [\mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0}, \lambda_\sigma^{-1} g_\sigma]) \text{d}u \text{d}\mu \text{d}\theta = 0, \\ \int B (u^2/2 + \mu B) (\tau_\sigma \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\lambda_\sigma^{-1} g_\sigma, \lambda_{\sigma'}^{-1} g_{\sigma'}] \\ + \tau_{\sigma'} \mathcal{T}_{\sigma',0}^* C_{\sigma'\sigma} [\lambda_{\sigma'}^{-1} g_{\sigma'}, \lambda_\sigma^{-1} g_\sigma]) \text{d}u \text{d}\mu \text{d}\theta = 0, \\ \int B (u^2/2 + \mu B) (\tau_\sigma \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\lambda_\sigma^{-2} g_\sigma, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0}] \\ + \tau_{\sigma'} \mathcal{T}_{\sigma',0}^* C_{\sigma'\sigma} [\mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0}, \lambda_\sigma^{-2} g_\sigma]) \text{d}u \text{d}\mu \text{d}\theta = 0, \end{aligned} \quad (139)$$

where $g_\sigma(\mathbf{r}, \mathbf{v}, t)$ and $g_{\sigma'}(\mathbf{r}, \mathbf{v}, t)$ are just placeholders for the functions that appear in the collision operators. We have not written these functions explicitly because their particular form is unimportant for the cancellations. To show that the summation of collision operators in (138) gives these binomials it is important to keep track of the factors λ_σ^{-1} that multiply the collision operator arguments.

Equation (137) can be simplified by further cancellations. Appendix L contains the proof that

$$\begin{aligned} - \left\langle \sum_\sigma \int B \left[F_{\sigma 1}^{\text{sw}} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right. \right. \right. \\ \left. \left. + \frac{\mu}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right. \right. \\ \left. \left. + \frac{u^2}{B} [\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right) \right]^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta \right\rangle_\psi \\ + \left\langle \sum_{\sigma, \sigma'} \frac{\tau_\sigma}{\lambda_\sigma^2} \int B (u^2/2 + \mu B) \left[\left\langle \mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{(1)\text{sw}} \right\rangle \right]^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta \right\rangle_\psi \\ = O(\epsilon_s), \end{aligned} \quad (140)$$

and therefore the final expression for the total energy transport equation is

$$\partial_{\epsilon_s^2 t} \left(\sum_\sigma \frac{3}{2} n_\sigma(\psi, t) T_\sigma(\psi, t) \right) =$$

$$\begin{aligned}
& \frac{1}{V'(\psi)} \partial_\psi \left\langle V'(\psi) \int (u^2/2 + \mu B) \sum_\sigma \left\{ \right. \right. \\
& \left. \left[F_{\sigma 1}^{\text{sw}} (\nabla_{\mathbf{r}_\perp/\epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \psi \right]^{\text{lw}} \right. \\
& \left. \left. + \frac{B}{Z_\sigma \lambda_\sigma} \left\langle \left(\frac{Iu}{B} + \boldsymbol{\rho} \cdot \nabla_{\mathbf{r}} \psi \right) \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(1)\text{lw}} \right\rangle \right\} \text{d}u \text{d}\mu \text{d}\theta \right\rangle_\psi. \quad (141)
\end{aligned}$$

5.3. Time evolution of the lowest-order quasineutrality condition: intrinsic ambipolarity of the turbulent tokamak

The zeroth-order piece of the long-wavelength quasineutrality equation imposes the well-known condition on the lowest order particle densities, equation (122):

$$\sum_\sigma Z_\sigma n_\sigma(\psi, t) = 0. \quad (142)$$

On the other hand, we have obtained as a solvability condition of the long-wavelength second-order Fokker-Planck equation a time evolution equation for each function n_σ , (134). Thus, we can deduce a time evolution equation for $\sum_\sigma Z_\sigma n_\sigma$. It is important to find out whether (142) is automatically preserved by the time evolution or, on the contrary, its preservation implies additional constraints on low-order quantities. In principle, it might have happened that imposing $\partial_t \sum_\sigma Z_\sigma n_\sigma = 0$ implied a new equation involving the long-wavelength radial electric field. In this subsection we show that this is not the case in a tokamak.

The contribution of the last term in (134) to $\partial_t \sum_\sigma Z_\sigma n_\sigma$ vanishes due to the momentum conservation properties of the collision operator (see (138)). As a result, we recover that, in neoclassical theory, $\partial_t \sum_\sigma Z_\sigma n_\sigma \equiv 0$. The contribution of the first term on the right side of (134) to $\partial_t \sum_\sigma Z_\sigma n_\sigma$ also vanishes,

$$\sum_\sigma Z_\sigma \left\langle \int B \left[F_{\sigma 1}^{\text{sw}} (\nabla_{\mathbf{r}_\perp/\epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \psi \right]^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta \right\rangle_\psi = 0. \quad (143)$$

To prove this property we need the short-wavelength quasineutrality equation to first order in the expansion parameter, given in (110). We repeat it here for convenience:

$$\begin{aligned}
& \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma} \int B \left[-Z_\sigma \lambda_\sigma \tilde{\phi}_{\sigma 1}^{\text{sw}}(\mathbf{r} - \epsilon_\sigma \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), \mu, \theta, t) \frac{F_{\sigma 0}(\mathbf{r}, u, \mu, t)}{T_\sigma(\mathbf{r}, t)} \right. \\
& \left. + F_{\sigma 1}^{\text{sw}}(\mathbf{r} - \epsilon_\sigma \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), u, \mu, t) \right] \text{d}u \text{d}\mu \text{d}\theta = 0. \quad (144)
\end{aligned}$$

Acting on (144) with $\varphi_1^{\text{sw}}(\mathbf{r}, t) \nabla_{\mathbf{r}_\perp/\epsilon_s}$, taking the coarse-grain average, and observing that

$$\varphi_1^{\text{sw}}(\mathbf{r}, t) = \phi_{\sigma 1}^{\text{sw}}(\mathbf{r} - \epsilon_\sigma \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), \mu, \theta, t) + O(\epsilon_\sigma), \quad (145)$$

we obtain

$$\sum_\sigma Z_\sigma \int B \left[\phi_{\sigma 1}^{\text{sw}} \nabla_{\mathbf{r}_\perp/\epsilon_\sigma} \left(-\frac{Z_\sigma \lambda_\sigma \tilde{\phi}_{\sigma 1}^{\text{sw}}}{T_\sigma} F_{\sigma 0} + F_{\sigma 1}^{\text{sw}} \right) \right]^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta = O(\epsilon_\sigma). \quad (146)$$

In this expression the functions $\phi_{\sigma 1}^{\text{sw}}$ and $F_{\sigma 1}^{\text{sw}}$ are evaluated at $\mathbf{R} = \mathbf{r} - \epsilon_\sigma \boldsymbol{\rho}(\mathbf{r}, \mu, \theta)$, but after the coarse grain average we can Taylor expand and, to lowest order, they can be evaluated at $\mathbf{R} = \mathbf{r}$. This leads to

$$\begin{aligned}
& - \sum_{\sigma} Z_{\sigma} \int B \nabla_{\mathbf{r}_{\perp}/\epsilon_{\sigma}} \left[\frac{Z_{\sigma} \lambda_{\sigma} (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2}{2T_{\sigma}} F_{\sigma 0} \right]^{\text{lw}} d\mu d\mu d\theta \\
& + \sum_{\sigma} Z_{\sigma} \int B [F_{\sigma 1}^{\text{sw}} \nabla_{\mathbf{r}_{\perp}/\epsilon_{\sigma}} \langle \phi_{\sigma 1} \rangle^{\text{sw}}]^{\text{lw}} d\mu d\mu d\theta = O(\epsilon_{\sigma}).
\end{aligned} \tag{147}$$

The fact that

$$\nabla_{\mathbf{r}_{\perp}/\epsilon_{\sigma}} g^{\text{lw}} = O(\epsilon_{\sigma}) \tag{148}$$

whenever $g = O(1)$ implies

$$\sum_{\sigma} Z_{\sigma} \int B [F_{\sigma 1}^{\text{sw}} \nabla_{\mathbf{r}_{\perp}/\epsilon_{\sigma}} \langle \phi_{\sigma 1} \rangle^{\text{sw}}]^{\text{lw}} d\mu d\mu d\theta = O(\epsilon_{\sigma}), \tag{149}$$

whence we immediately infer equation (143), giving

$$\partial_t \sum_{\sigma} Z_{\sigma} n_{\sigma} \equiv 0, \tag{150}$$

identically, at the relevant order. Consequently, we have proven that the well-known neoclassical intrinsic ambipolarity property of the tokamak still holds in gyrokinetic theory.

6. Discussion of results and conclusions

At the moment, the problem of extending the standard set of gyrokinetic equations, and therefore computer simulations, to transport time scales is an active research topic. Focusing on toroidal angular momentum transport in tokamaks in electrostatic gyrokinetics, the issue has been recently raised by Parra and Catto [12, 18, 19, 20, 21, 22]; they argue that calculating momentum transport correctly requires knowledge of the distribution function and electrostatic potential up to second order in the expansion parameter, the gyroradius over the macroscopic length scale. An intimately related result of this series of works is that in a tokamak the system consisting of second-order Fokker-Planck and quasineutrality equations does not determine the long-wavelength radial electric field. A method to correctly compute the radial transport of toroidal angular momentum (and therefore the radial electric field) when the second-order pieces of the distribution function and the electrostatic potential are known is given in reference [22, 24].

Using the recent derivation of the second-order gyrokinetic equations [16] in general magnetic geometry, we have worked out the long-wavelength limit of the Fokker-Planck and quasineutrality equations in a tokamak, a necessary first step towards the formulation of a set of equations to compute the radial transport of toroidal angular momentum without having to resort to subsidiary expansions such as the expansion in

$B_p/B \ll 1$ of references [22, 24]. Specifically, we have obtained (see the main text for notation and details):

- (i) The long-wavelength Fokker-Planck equations to second order, (105) and (119), that give $G_{\sigma 1}^{\text{lw}}$ and $G_{\sigma 2}^{\text{lw}}$, and therefore the long-wavelength component of the distribution functions.
- (ii) The quasineutrality equation up to second-order (122), (130), and (131), that determines the first and second-order pieces of the long-wavelength poloidal electric field. Equivalently, and under conditions (132) and (133), the quasineutrality equation determines φ_1^{lw} and φ_2^{lw} .
- (iii) Transport equations for density (134) and energy (141).
- (iv) Equations (109) and (110), that give the short-wavelength component of the distribution functions, $F_{\sigma 1}^{\text{sw}}$, and electrostatic potential, $\phi_{\sigma 1}^{\text{sw}}$. They are needed because they enter equation (119).

In order to provide a model for toroidal angular momentum transport in tokamaks one still needs to derive explicit equations for the short-wavelength components of the distribution functions and electrostatic potential to second order. This will be the subject of a future publication.

In addition, in this paper, we have given a complete proof that the long-wavelength tokamak radial electric field cannot be determined by simply using Fokker-Planck and quasineutrality equations accurate to second order in the gyrokinetic expansion parameter. In other words, we have proven that gyrokinetics does not spoil the well-known neoclassical intrinsic ambipolarity property of the tokamak.

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Appendix A. Gyrokinetic equations of motion

Here, the gyrokinetic equations of motion corresponding to the Poisson bracket (53) and the gyrokinetic Hamiltonian given in equations (58), (59), and (60), are explicitly written:

$$\begin{aligned} \frac{d\mathbf{R}}{dt} = \{\mathbf{R}, \overline{H}_\sigma\}_{\mathbf{Z}} = & \\ & \left(u + Z_\sigma \lambda_\sigma \epsilon_\sigma^2 \partial_u \Psi_{\phi B, \sigma} + \epsilon_\sigma^2 \partial_u \Psi_{B, \phi} \right) \frac{\mathbf{B}_\sigma^*}{B_{\parallel, \sigma}^*} \\ & + \frac{1}{B_{\parallel, \sigma}^*} \hat{\mathbf{b}} \times \left(\epsilon_\sigma \mu \nabla_{\mathbf{R}} B + Z_\sigma \lambda_\sigma \epsilon_\sigma \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_\sigma \rangle \right. \\ & \left. + Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma^2 \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \Psi_{\phi, \sigma} + Z_\sigma \lambda_\sigma \epsilon_\sigma^2 \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \Psi_{\phi B, \sigma} \right) \end{aligned}$$

$$+ Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{\phi, \sigma} + Z_\sigma \lambda_\sigma \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{\phi B, \sigma} + \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{B, \sigma} \Big), \quad (\text{A.1})$$

$$\begin{aligned} \frac{du}{dt} = \{u, \overline{H}_\sigma\}_{\mathbf{Z}} = & - \frac{\mu}{B_{||, \sigma}^*} \mathbf{B}_\sigma^* \cdot \nabla_{\mathbf{R}} B - Z_\sigma \lambda_\sigma \epsilon_\sigma \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_\sigma \rangle \\ & - Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma^2 \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Psi_{\phi, \sigma} - Z_\sigma \lambda_\sigma \epsilon_\sigma^2 \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Psi_{\phi B, \sigma} \\ & - \epsilon_\sigma^2 \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Psi_{B, \sigma} - \frac{1}{B_{||, \sigma}^*} \left[u \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right. \\ & \left. - \epsilon_\sigma \mu (\nabla_{\mathbf{R}} \times \mathbf{K})_\perp \right] \cdot \left(Z_\sigma \lambda_\sigma \epsilon_\sigma \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_\sigma \rangle \right. \\ & + Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma^2 \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \Psi_{\phi, \sigma} + Z_\sigma \lambda_\sigma \epsilon_\sigma^2 \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \Psi_{\phi B, \sigma} \\ & \left. + Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{\phi, \sigma} + Z_\sigma \lambda_\sigma \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{\phi B, \sigma} + \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{B, \sigma} \right), \quad (\text{A.2}) \end{aligned}$$

$$\frac{d\mu}{dt} = \{\mu, \overline{H}_\sigma\}_{\mathbf{Z}} = 0, \quad (\text{A.3})$$

$$\begin{aligned} \frac{d\theta}{dt} = \{\theta, \overline{H}_\sigma\}_{\mathbf{Z}} = & - \frac{1}{\epsilon_\sigma} B - Z_\sigma \lambda_\sigma \partial_\mu \langle \phi_\sigma \rangle - Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma \partial_\mu \Psi_{\phi, \sigma} \\ & - Z_\sigma \lambda_\sigma \epsilon_\sigma \partial_\mu \Psi_{\phi B, \sigma} - \epsilon_\sigma \partial_\mu \Psi_{B, \sigma} \\ & - \frac{\mathbf{B}_\sigma^* \cdot \mathbf{K}}{B_{||, \sigma}^*} \left(u + Z_\sigma \lambda_\sigma \epsilon_\sigma^2 \partial_u \Psi_{\phi B, \sigma} + \epsilon_\sigma^2 \partial_u \Psi_{B, \sigma} \right) \\ & - \frac{1}{B_{||, \sigma}^*} (\mathbf{K} \times \hat{\mathbf{b}}) \cdot \left(\epsilon_\sigma \mu \nabla_{\mathbf{R}} B + Z_\sigma \lambda_\sigma \epsilon_\sigma \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_\sigma \rangle \right. \\ & + Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma^2 \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \Psi_{\phi, \sigma} + Z_\sigma \lambda_\sigma \epsilon_\sigma^2 \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \Psi_{\phi B, \sigma} \\ & \left. + Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{\phi, \sigma} + Z_\sigma \lambda_\sigma \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{\phi B, \sigma} + \epsilon_\sigma^3 \nabla_{\mathbf{R}} \Psi_{B, \sigma} \right). \quad (\text{A.4}) \end{aligned}$$

Appendix B. Some basic properties of the collision operator

We recall (see, for example, reference [41]) that the collision operator (3) satisfies, for every σ, σ' , the conservation properties

$$\begin{aligned} \int C_{\sigma\sigma'} d^3v &= 0 \\ \int m_\sigma \mathbf{v} C_{\sigma\sigma'} d^3v &= - \int m_{\sigma'} \mathbf{v} C_{\sigma'\sigma} d^3v \\ \int \frac{1}{2} m_\sigma \mathbf{v}^2 C_{\sigma\sigma'} d^3v &= - \int \frac{1}{2} m_{\sigma'} \mathbf{v}^2 C_{\sigma'\sigma} d^3v, \quad (\text{B.1}) \end{aligned}$$

and the statistical equilibrium condition

$$C_{\sigma\sigma'}[f_{M\sigma}, f_{M\sigma'}] = 0 \quad (\text{B.2})$$

when both distribution functions are Maxwellian,

$$\begin{aligned} f_{M\sigma}(\mathbf{r}, \mathbf{v}) &= n_\sigma(\mathbf{r}) \left(\frac{m_\sigma}{2\pi T_\sigma(\mathbf{r})} \right)^{3/2} \exp \left(-\frac{m_\sigma \mathbf{v}^2}{2T_\sigma(\mathbf{r})} \right), \\ f_{M\sigma'}(\mathbf{r}, \mathbf{v}) &= n_{\sigma'}(\mathbf{r}) \left(\frac{m_{\sigma'}}{2\pi T_{\sigma'}(\mathbf{r})} \right)^{3/2} \exp \left(-\frac{m_{\sigma'} \mathbf{v}^2}{2T_{\sigma'}(\mathbf{r})} \right), \end{aligned} \quad (\text{B.3})$$

with $T_\sigma(\mathbf{r}) = T_{\sigma'}(\mathbf{r})$ at every point. These are the only solutions to the equations (B.2). The easiest way to see this is noting that the entropy production,

$$-\sum_{\sigma, \sigma'} \int \ln f_\sigma C_{\sigma\sigma'}[f_\sigma, f_{\sigma'}] d^3v, \quad (\text{B.4})$$

is non-negative and vanishes only when f_σ and $f_{\sigma'}$ are Maxwellians with the same temperature.

Another well-known property, derived from (B.2), is

$$C_{\sigma\sigma'} \left[\frac{m_\sigma}{T_\sigma} \mathbf{v} f_{M\sigma}, f_{M\sigma'} \right] + C_{\sigma\sigma'} \left[f_{M\sigma}, \frac{m_{\sigma'}}{T_{\sigma'}} \mathbf{v} f_{M\sigma'} \right] \equiv 0. \quad (\text{B.5})$$

This property implies that displacing both Maxwellians by the same average velocity gives another solution of (B.2).

It is useful to have the explicit translation of these properties into our non-dimensional variables. With the definition (15) we have

$$\begin{aligned} \int \underline{C}_{\sigma\sigma'} d^3\underline{v} &= 0 \\ \int \underline{\mathbf{v}} \underline{C}_{\sigma\sigma'} d^3\underline{v} &= - \int \underline{\mathbf{v}} \underline{C}_{\sigma'\sigma} d^3\underline{v} \\ \int \frac{1}{2} \tau_\sigma \underline{\mathbf{v}}^2 \underline{C}_{\sigma\sigma'} d^3\underline{v} &= - \int \frac{1}{2} \tau_{\sigma'} \underline{\mathbf{v}}^2 \underline{C}_{\sigma'\sigma} d^3\underline{v}. \end{aligned} \quad (\text{B.6})$$

Also,

$$\underline{C}_{\sigma\sigma'}[\underline{f}_{M\sigma}, \underline{f}_{M\sigma'}] \equiv 0 \quad (\text{B.7})$$

when

$$\begin{aligned} \underline{f}_{M\sigma}(\underline{\mathbf{r}}, \underline{\mathbf{v}}) &= \frac{n_\sigma(\underline{\mathbf{r}})}{(2\pi \underline{T}_\sigma(\underline{\mathbf{r}}))^{3/2}} \exp \left(-\frac{\underline{\mathbf{v}}^2}{2\underline{T}_\sigma(\underline{\mathbf{r}})} \right), \\ \underline{f}_{M\sigma'}(\underline{\mathbf{r}}, \underline{\mathbf{v}}) &= \frac{n_{\sigma'}(\underline{\mathbf{r}})}{(2\pi \underline{T}_{\sigma'}(\underline{\mathbf{r}}))^{3/2}} \exp \left(-\frac{\underline{\mathbf{v}}^2}{2\underline{T}_{\sigma'}(\underline{\mathbf{r}})} \right), \end{aligned} \quad (\text{B.8})$$

and $\underline{T}_\sigma(\underline{\mathbf{r}}) = \underline{T}_{\sigma'}(\underline{\mathbf{r}})$ at every point. Finally,

$$\underline{C}_{\sigma\sigma'} \left[\frac{1}{\tau_\sigma \underline{T}_\sigma} \underline{\mathbf{v}} \underline{f}_{M\sigma}, \underline{f}_{M\sigma'} \right] + \underline{C}_{\sigma\sigma'} \left[\underline{f}_{M\sigma}, \frac{1}{\tau_{\sigma'} \underline{T}_{\sigma'}} \underline{\mathbf{v}} \underline{f}_{M\sigma'} \right] \equiv 0. \quad (\text{B.9})$$

Appendix C. Gyrokinetic transformation to first order

In this appendix we provide explicit expressions for the gyrokinetic transformation $(\mathbf{r}, \mathbf{v}) = \mathcal{T}_\sigma(\mathbf{R}, u, \mu, \theta, t)$ to order ϵ_σ . Define

$$v_{||} := \mathbf{v} \cdot \hat{\mathbf{b}}(\mathbf{r}), \quad (\text{C.1})$$

$$\mu_0 := \frac{(\mathbf{v} - v_{||}\hat{\mathbf{b}}(\mathbf{r}))^2}{2B(\mathbf{r})}, \quad (\text{C.2})$$

$$\theta_0 := \arctan\left(\frac{\mathbf{v} \cdot \hat{\mathbf{e}}_2(\mathbf{r})}{\mathbf{v} \cdot \hat{\mathbf{e}}_1(\mathbf{r})}\right), \quad (\text{C.3})$$

and let us compute $(\mathbf{r}, v_{||}, \mu_0, \theta_0)$ as a function of $(\mathbf{R}, u, \mu, \theta)$ to first order in ϵ_σ . From the definition (35) we find $(\mathbf{r}, v_{||}, \mu_0, \theta_0)$ as a function of $(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g)$:

$$\begin{aligned} \mathbf{r} &= \mathbf{R}_g + \epsilon_\sigma \boldsymbol{\rho}_g, \\ v_{||} &= v_{||g} + \epsilon_\sigma B_g(\boldsymbol{\rho}_g \times \hat{\mathbf{b}}_g) \boldsymbol{\rho}_g : \nabla_{\mathbf{R}_g} \hat{\mathbf{b}}_g + O(\epsilon_\sigma^2), \\ \mu_0 &= \mu_g - \epsilon_\sigma \left(\frac{\mu_g}{B_g} \boldsymbol{\rho}_g \cdot \nabla_{\mathbf{R}_g} B_g + v_{||g}(\boldsymbol{\rho}_g \times \hat{\mathbf{b}}_g) \boldsymbol{\rho}_g : \nabla_{\mathbf{R}_g} \hat{\mathbf{b}}_g \right) \\ &\quad + O(\epsilon_\sigma^2), \\ \theta_0 &= \theta_g + \epsilon_\sigma \left(\boldsymbol{\rho}_g \cdot \nabla_{\mathbf{R}_g} \hat{\mathbf{e}}_{2g} \cdot \hat{\mathbf{e}}_{1g} - \frac{v_{||g}}{2\mu_g} \boldsymbol{\rho}_g \cdot \nabla_{\mathbf{R}_g} \hat{\mathbf{b}}_g \cdot \boldsymbol{\rho}_g \right) \\ &\quad + O(\epsilon_\sigma^2), \end{aligned} \quad (\text{C.4})$$

where a subindex g stresses that the quantity is evaluated at $(\mathbf{R}_g, v_{||g}, \mu_g, \theta_g)$. Using (38), (68), (69), (70), and the identities

$$\begin{aligned} (\boldsymbol{\rho} \times \hat{\mathbf{b}}) \boldsymbol{\rho} : \nabla_{\mathbf{R}} \hat{\mathbf{b}} &= \boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) : \nabla_{\mathbf{R}} \hat{\mathbf{b}} - \frac{2\mu}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}, \\ \boldsymbol{\rho} \boldsymbol{\rho} + (\boldsymbol{\rho} \times \hat{\mathbf{b}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}) &= \frac{2\mu}{B} (\vec{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}), \end{aligned} \quad (\text{C.5})$$

we arrive at

$$\begin{aligned} \mathbf{r} &= \mathbf{R} + \epsilon_\sigma \boldsymbol{\rho} + O(\epsilon_\sigma^2), \\ v_{||} &= u + \epsilon_\sigma \hat{u}_1 + O(\epsilon_\sigma^2), \\ \mu_0 &= \mu + \epsilon_\sigma \hat{\mu}_1 + O(\epsilon_\sigma^2), \\ \theta_0 &= \theta + \epsilon_\sigma \hat{\theta}_1 + O(\epsilon_\sigma^2), \end{aligned} \quad (\text{C.6})$$

where

$$\begin{aligned} \hat{u}_1 &= u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} + \frac{B}{4} [\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}}) \boldsymbol{\rho}] : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \\ &\quad - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}, \\ \hat{\mu}_1 &= -\frac{\mu}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B - \frac{u}{4} (\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}}) \boldsymbol{\rho}) : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \end{aligned}$$

$$\begin{aligned}
& + \frac{u\mu}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} - \frac{u^2}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} - \frac{Z_\sigma \lambda_\sigma}{B} \tilde{\phi}_{\sigma 1}, \\
\hat{\theta}_1 &= (\boldsymbol{\rho} \times \hat{\mathbf{b}}) \cdot \left(\nabla_{\mathbf{R}} \ln B + \frac{u^2}{2\mu B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \right. \\
& \quad \left. - \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 \right) - \frac{u}{8\mu} \left(\boldsymbol{\rho} \boldsymbol{\rho} - (\boldsymbol{\rho} \times \hat{\mathbf{b}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}) \right) : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \\
& \quad + \frac{u}{2B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B + \frac{Z_\sigma \lambda_\sigma}{B} \partial_\mu \tilde{\Phi}_{\sigma 1}. \tag{C.7}
\end{aligned}$$

It is useful to have the long-wavelength limit of the previous expressions at hand. Employing (49), and (50) we get:

$$\begin{aligned}
\hat{u}_1^{\text{lw}} &= \hat{u}_1 \\
\hat{\mu}_1^{\text{lw}} &= -\frac{\mu}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B - \frac{u}{4} \left(\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}})\boldsymbol{\rho} \right) : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \\
& \quad + \frac{u\mu}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} - \frac{u^2}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} - \frac{Z_\sigma}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \varphi_0, \\
\hat{\theta}_1^{\text{lw}} &= (\boldsymbol{\rho} \times \hat{\mathbf{b}}) \cdot \left(\nabla_{\mathbf{R}} \ln B + \frac{u^2}{2\mu B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \right. \\
& \quad \left. - \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 \right) - \frac{u}{8\mu} \left(\boldsymbol{\rho} \boldsymbol{\rho} - (\boldsymbol{\rho} \times \hat{\mathbf{b}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}) \right) : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \\
& \quad + \frac{u}{2B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B + \frac{Z_\sigma}{2\mu B} (\boldsymbol{\rho} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \varphi_0. \tag{C.8}
\end{aligned}$$

Next, we proceed to calculate the long-wavelength limit of $\mathcal{T}_\sigma^{-1*} F_{\sigma 0}$ to first order in ϵ_σ , needed to write (96) in Section 3.3. Inverting (C.6) to first order, and recalling (C.8) and the relations $\partial_u F_{\sigma 0} = -(u/T_\sigma) F_{\sigma 0}$, $\partial_\mu F_{\sigma 0} = -(B/T_\sigma) F_{\sigma 0}$, one finds that

$$\begin{aligned}
[\mathcal{T}_\sigma^{-1*} F_{\sigma 0}]^{\text{lw}} &= \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} + \frac{\epsilon_\sigma}{T_\sigma} \left[\mathbf{v} \cdot \mathbf{V}_\sigma^p + \left(\frac{v^2}{2T_\sigma} - \frac{5}{2} \right) \mathbf{v} \cdot \mathbf{V}_\sigma^T \right. \\
& \quad \left. + \frac{Z_\sigma}{B} \mathbf{v} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{r}} \varphi_0) \right] \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \tag{C.9}
\end{aligned}$$

with \mathbf{V}_σ^p and \mathbf{V}_σ^T defined in (101).

Appendix D. Calculations for the Fokker-Planck equation to $O(\epsilon_\sigma)$

In what follows we detail the calculations that recast (103) into (105) when the magnetic field has the form (104). First, rewrite (103) as

$$\begin{aligned}
& \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) F_{\sigma 1}^{\text{lw}} \\
& + \left(\mathbf{v}_\kappa + \mathbf{v}_{\nabla B} + \mathbf{v}_{E,\sigma}^{(0)} \right) \cdot \left(\nabla_{\mathbf{R}} F_{\sigma 0} + \frac{\mu F_{\sigma 0}}{T_\sigma} \nabla_{\mathbf{R}} B \right. \\
& \quad \left. + \frac{Z_\sigma F_{\sigma 0}}{T_\sigma} \nabla_{\mathbf{R}} \varphi_0 \right) + \frac{Z_\sigma \lambda_\sigma}{T_\sigma} u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} F_{\sigma 0}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 1}^{\text{lw}}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0}] \\
&+ \sum_{\sigma'} \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 1}^{\text{lw}}].
\end{aligned} \tag{D.1}$$

Denote by R the cylindrical coordinate giving the distance to the axis of the torus, and by $\hat{\boldsymbol{\zeta}}$ the unit vector in the toroidal direction. The identities

$$B^2 = \frac{I^2 + |\nabla_{\mathbf{R}} \psi|^2}{R^2}, \tag{D.2}$$

$$\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi = I \hat{\mathbf{b}} - R B \hat{\boldsymbol{\zeta}}, \tag{D.3}$$

$$\nabla_{\mathbf{R}} \cdot \hat{\boldsymbol{\zeta}} = 0, \tag{D.4}$$

$$\begin{aligned}
&[\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot \nabla_{\mathbf{R}} \psi = \\
&(\nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \psi = \nabla_{\mathbf{R}} \cdot (I \hat{\mathbf{b}}) = \mathbf{B} \cdot \nabla_{\mathbf{R}} \left(\frac{I}{B} \right),
\end{aligned} \tag{D.5}$$

$$\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \varphi_0 = \partial_\psi \varphi_0 (I \hat{\mathbf{b}} - R B \hat{\boldsymbol{\zeta}}), \tag{D.6}$$

and

$$(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B) \cdot \nabla_{\mathbf{R}} \psi = -I \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \tag{D.7}$$

are useful to write (D.1) as

$$\begin{aligned}
&\left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) F_{\sigma 1}^{\text{lw}} \\
&\left(u^2 \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \left(\frac{I}{B} \right) - \frac{I \mu}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \right) \left(\Upsilon_\sigma \right. \\
&\left. + \frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 \right) F_{\sigma 0} + \frac{Z_\sigma \lambda_\sigma u}{T_\sigma} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} F_{\sigma 0} \\
&= \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 1}^{\text{lw}}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0}] \\
&+ \sum_{\sigma'} \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 1}^{\text{lw}}],
\end{aligned} \tag{D.8}$$

where Υ_σ is defined in (107). Equation (D.8) is equivalent to

$$\begin{aligned}
&(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u) \left[F_{\sigma 1}^{\text{lw}} \right. \\
&\left. + \left\{ \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \varphi_1^{\text{lw}} + \frac{I u}{B} \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right) \right\} F_{\sigma 0} \right] \\
&= \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 1}^{\text{lw}}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0}] \\
&+ \sum_{\sigma'} \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 1}^{\text{lw}}].
\end{aligned} \tag{D.9}$$

The definition of the new function $G_{\sigma 1}^{\text{lw}}$ given in (106) seems appropriate. Employing that the collision operator vanishes when acting on Maxwellians with the same temperature, and using property (B.9), the dependence on φ_0 and φ_1^{lw} is removed and equation (105) is obtained.

Appendix E. Computation of the turbulent piece of the collision operator

We have to calculate $[\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{\text{sw}}]^{\text{lw}}$ appearing in (116) and this appendix is devoted to that end. Then,

$$\begin{aligned} [\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{\text{sw}}]^{\text{lw}} = & \left[\frac{Z_\sigma \lambda_\sigma}{B} \left(-\frac{1}{B} \left(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \tilde{\Phi}_\sigma^{\text{sw}} \right) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} - \tilde{\phi}_{\sigma 1}^{\text{sw}} \partial_\mu + \partial_\mu \tilde{\Phi}_{\sigma 1}^{\text{sw}} \partial_\theta \right) \right. \\ & \left\{ \mathcal{T}_{NP,\sigma}^* C_{\sigma\sigma'} \left[\mathbb{T}_{\sigma,0} F_{\sigma 1}^{\text{sw}} - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma,0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \right. \\ & + \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{NP,\sigma}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma',0} F_{\sigma' 1}^{\text{sw}} \right. \\ & \left. \left. \left. - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma',0} \tilde{\phi}_{\sigma' 1}^{\text{sw}} \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \right] \right\}^{\text{lw}}. \end{aligned} \quad (\text{E.1})$$

But the first term on the right side of (E.1) does not contribute in the long-wavelength limit because, for any $g = O(1)$,

$$\begin{aligned} \left[\left(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \tilde{\Phi}_\sigma^{\text{sw}} \right) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} g^{\text{sw}} \right]^{\text{lw}} = & -\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \times \left[g^{\text{sw}} \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \tilde{\Phi}_\sigma^{\text{sw}} \right]^{\text{lw}} = O(\epsilon_s). \end{aligned} \quad (\text{E.2})$$

Therefore,

$$\begin{aligned} [\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{\text{sw}}]^{\text{lw}} = & \left[\frac{Z_\sigma \lambda_\sigma}{B} \left(-\tilde{\phi}_{\sigma 1}^{\text{sw}} \partial_\mu + \partial_\mu \tilde{\Phi}_{\sigma 1}^{\text{sw}} \partial_\theta \right) \right. \\ & \left\{ \mathcal{T}_{NP,\sigma}^* C_{\sigma\sigma'} \left[\mathbb{T}_{\sigma,0} F_{\sigma 1}^{\text{sw}} - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma,0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \right. \\ & + \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{NP,\sigma}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma',0} F_{\sigma' 1}^{\text{sw}} \right. \\ & \left. \left. \left. - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma',0} \tilde{\phi}_{\sigma' 1}^{\text{sw}} \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \right] \right\}^{\text{lw}}. \end{aligned} \quad (\text{E.3})$$

As for its gyroaverage,

$$\begin{aligned} \left\langle [\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{\text{sw}}]^{\text{lw}} \right\rangle = & -\partial_\mu \left\langle \left[\frac{Z_\sigma \lambda_\sigma}{B} \tilde{\phi}_{\sigma 1}^{\text{sw}} \right. \right. \\ & \left. \left. \left\{ \mathcal{T}_{NP,\sigma}^* C_{\sigma\sigma'} \left[\mathbb{T}_{\sigma,0} F_{\sigma 1}^{\text{sw}} - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma,0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \right] \right\} \right\rangle \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{NP,\sigma}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma',0} F_{\sigma'1}^{\text{sw}} \right. \\
& \left. - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma',0} \tilde{\phi}_{\sigma'1}^{\text{sw}} \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right] \Bigg] \Bigg\rangle^{\text{lw}}. \tag{E.4}
\end{aligned}$$

In order to get the last expression we have integrated by parts in θ and μ .

Appendix F. Computation of the last term of (111)

$$\begin{aligned}
C_{\sigma\sigma'}^{(2)\text{lw}} = & C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 2}^{\text{lw}} + [\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 1}^{\text{lw}}]^{\text{lw}} \right. \\
& + [\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 1}^{\text{sw}}]^{\text{lw}} + [\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \Big] \\
& + \left(\frac{\lambda_\sigma}{\lambda_{\sigma'}} \right)^2 C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'2}^{\text{lw}} \right. \\
& + [\mathcal{T}_{\sigma',1}^{-1*} F_{\sigma'1}^{\text{lw}}]^{\text{lw}} + [\mathcal{T}_{\sigma',1}^{-1*} F_{\sigma'1}^{\text{sw}}]^{\text{lw}} + [\mathcal{T}_{\sigma',2}^{-1*} F_{\sigma'0}]^{\text{lw}} \Big] \\
& + \frac{\lambda_\sigma}{\lambda_{\sigma'}} C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 1}^{\text{lw}} + [\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 0}]^{\text{lw}}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'1}^{\text{lw}} + [\mathcal{T}_{\sigma',1}^{-1*} F_{\sigma'0}]^{\text{lw}} \right] \\
& + \frac{\lambda_\sigma}{\lambda_{\sigma'}} \left[C_{\sigma\sigma'} \left[\mathbb{T}_{\sigma,0} F_{\sigma 1}^{\text{sw}} - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma,0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma',0} F_{\sigma'1}^{\text{sw}} \right. \right. \\
& \left. \left. - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma',0} \tilde{\phi}_{\sigma'1}^{\text{sw}} \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right] \right]^{\text{lw}}, \tag{F.1}
\end{aligned}$$

with

$$\begin{aligned}
[\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 1}^{\text{lw}}]^{\text{lw}} = & -\mathcal{T}_{\sigma,0}^{-1*} \left\{ \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \right. \\
& + \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} + \frac{B}{4} [\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}})\boldsymbol{\rho}] : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \right. \\
& \left. - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right) \partial_u + \left(-\frac{\mu}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B \right. \\
& - \frac{u}{4} [\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}})\boldsymbol{\rho}] : \nabla_{\mathbf{R}} \hat{\mathbf{b}} + \frac{u\mu}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \\
& \left. \left. - \frac{u^2}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} - \frac{Z_\sigma}{B} \boldsymbol{\rho} \cdot \nabla \varphi_0 \right) \partial_\mu \right\} F_{\sigma 1}^{\text{lw}}, \tag{F.2}
\end{aligned}$$

$$\begin{aligned}
[\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 1}^{\text{sw}}]^{\text{lw}} = & -\mathcal{T}_{\sigma,0}^{-1*} \left[\left(\frac{Z_\sigma \lambda_\sigma}{B^2} (\nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \tilde{\Phi}_{\sigma 1}^{\text{sw}} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \right. \right. \\
& \left. \left. - \frac{Z_\sigma \lambda_\sigma \tilde{\phi}_{\sigma 1}^{\text{sw}}}{B} \frac{\partial}{\partial \mu} \right) F_{\sigma 1}^{\text{sw}} \right]^{\text{lw}}. \tag{F.3}
\end{aligned}$$

Here to obtain (F.2) and (F.3) we have used the results in Appendix C. The term $[\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}}$ is calculated in Appendix G. Now, let us write the gyroaverage of (F.1):

$$\left\langle \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(2)\text{lw}} \right\rangle = \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} \langle F_{\sigma 2}^{\text{lw}} \rangle + \mathcal{T}_{\sigma,0}^{-1*} \langle \mathcal{T}_{\sigma,1}^{-1*} [\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 1}^{\text{lw}}]^{\text{lw}} \rangle \right]$$

$$\begin{aligned}
& + \mathcal{T}_{\sigma,0}^{-1*} \langle \mathcal{T}_{\sigma,0}^* [\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}} \rangle, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \rangle + \left(\frac{\lambda_\sigma}{\lambda_{\sigma'}} \right)^2 \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \right. \\
& \left. \mathcal{T}_{\sigma',0}^{-1*} \langle F_{\sigma'2}^{\text{lw}} \rangle + \mathcal{T}_{\sigma',0}^{-1*} \langle \mathcal{T}_{\sigma',0}^* [\mathcal{T}_{\sigma',1}^{-1*} F_{\sigma'1}^{\text{lw}}]^{\text{lw}} \rangle + \mathcal{T}_{\sigma',0}^{-1*} \langle \mathcal{T}_{\sigma',0}^* [\mathcal{T}_{\sigma',2}^{-1*} F_{\sigma'0}]^{\text{lw}} \rangle \right] \\
& + \frac{\lambda_\sigma}{\lambda_{\sigma'}} \langle \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} [\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 1}^{\text{lw}} + [\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 0}]^{\text{lw}}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'1}^{\text{lw}} + [\mathcal{T}_{\sigma',1}^{-1*} F_{\sigma'0}]^{\text{lw}}] \rangle \\
& + \frac{\lambda_\sigma}{\lambda_{\sigma'}} \left\langle \mathcal{T}_{\sigma,0}^* \left[C_{\sigma\sigma'} \left[\mathbb{T}_{\sigma,0} F_{\sigma 1}^{\text{sw}} - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma,0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma',0} F_{\sigma'1}^{\text{sw}} \right. \right. \right. \\
& \left. \left. \left. - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma',0} \tilde{\phi}_{\sigma'1}^{\text{sw}} \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right] \right]^{\text{lw}} \right\rangle, \tag{F.4}
\end{aligned}$$

where we have used that $\langle \mathcal{T}_{\sigma,0}^* [\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 1}^{\text{sw}}]^{\text{lw}} \rangle = 0$. Here,

$$\langle \mathcal{T}_{\sigma,0}^* [\mathcal{T}_{\sigma,1}^{-1*} F_{\sigma 1}^{\text{sw}}]^{\text{lw}} \rangle = \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \left(\partial_u - \frac{u}{B} \partial_\mu \right) F_{\sigma 1}^{\text{sw}}, \tag{F.5}$$

and

$$\begin{aligned}
& \langle \mathcal{T}_{\sigma,0}^* [\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}} \rangle = \\
& \frac{\mu}{2B} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \left[\nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \ln n_\sigma + \left(\frac{u^2/2 + \mu B}{T_\sigma} - \frac{3}{2} \right) \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \ln T_\sigma \right] F_{\sigma 0} \\
& - \frac{\mu}{B} \frac{Z_\sigma}{T_\sigma^2} \nabla_{\mathbf{R}} \varphi_0 \cdot \nabla_{\mathbf{R}} T_\sigma F_{\sigma 0} - \frac{\mu}{2B} \frac{u^2/2 + \mu B}{T_\sigma^3} |\nabla_{\mathbf{R}} T_\sigma|^2 F_{\sigma 0} \\
& + \frac{\mu}{2B} \left| \frac{\nabla_{\mathbf{R}} n_\sigma}{n_\sigma} + \frac{Z_\sigma \nabla_{\mathbf{R}} \varphi_0}{T_\sigma} + \left(\frac{u^2/2 + \mu B}{T_\sigma} - \frac{3}{2} \right) \frac{\nabla_{\mathbf{R}} T_\sigma}{T_\sigma} \right|^2 F_{\sigma 0} \\
& - \frac{\mu}{2B^2} \nabla_{\mathbf{R}_\perp} B \cdot \left(\frac{\nabla_{\mathbf{R}} n_\sigma}{n_\sigma} + \frac{Z_\sigma \nabla_{\mathbf{R}} \varphi_0}{T_\sigma} + \left(\frac{u^2/2 + \mu B}{T_\sigma} - \frac{3}{2} \right) \frac{\nabla_{\mathbf{R}} T_\sigma}{T_\sigma} \right) F_{\sigma 0} \\
& + \frac{Z_\sigma^2 \lambda_\sigma^2}{2T_\sigma^2} \left[\left\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right\rangle \right]^{\text{lw}} F_{\sigma 0} + \frac{1}{T_\sigma} \left[- \frac{Z_\sigma^2}{2B^2} |\nabla_{\mathbf{R}} \varphi_0|^2 \right. \\
& - \frac{Z_\sigma^2 \lambda_\sigma^2}{2B} \partial_\mu \left[\left\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right\rangle \right]^{\text{lw}} - \frac{3Z_\sigma \mu}{2B^2} \nabla_{\mathbf{R}_\perp} B \cdot \nabla_{\mathbf{R}} \varphi_0 \\
& - \frac{Z_\sigma u^2}{B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_0 + \Psi_{B,\sigma} \\
& \left. + \frac{Z_\sigma \mu}{B} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \varphi_0 \right] F_{\sigma 0}. \tag{F.6}
\end{aligned}$$

This last result has been obtained by gyroaveraging (G.19).

Appendix G. Second-order inverse transformation of a Maxwellian

The calculation of $C_{\sigma\sigma'}^{(2)\text{lw}}$ in Appendix F requires $[\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}}$. We start by using that $F_{\sigma 0}$ is a Maxwellian that depends on \mathbf{R} and $u^2/2 + \mu B(\mathbf{R})$, giving

$$\begin{aligned}
& [\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}} = \\
& \frac{1}{2B^2} (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) : \left[\nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \ln n_\sigma + \left(\frac{v^2}{2T_\sigma} - \frac{3}{2} \right) \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \ln T_\sigma \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{v^2}{2T_\sigma^3} \nabla_{\mathbf{r}} T_\sigma \nabla_{\mathbf{r}} T_\sigma + \left(\frac{\nabla_{\mathbf{r}} n_\sigma}{n_\sigma} + \left(\frac{v^2}{2T_\sigma} - \frac{3}{2} \right) \frac{\nabla_{\mathbf{r}} T_\sigma}{T_\sigma} \right) \left(\frac{\nabla_{\mathbf{r}} n_\sigma}{n_\sigma} \right. \\
& \left. + \left(\frac{v^2}{2T_\sigma} - \frac{3}{2} \right) \frac{\nabla_{\mathbf{r}} T_\sigma}{T_\sigma} \right) \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \\
& + \mathbf{R}_{02}^{\text{lw}} \cdot \left(\frac{\nabla_{\mathbf{r}} n_\sigma}{n_\sigma} + \left(\frac{v^2}{2T_\sigma} - \frac{3}{2} \right) \frac{\nabla_{\mathbf{r}} T_\sigma}{T_\sigma} \right) \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \\
& - \frac{1}{B} H_{01}^{\text{lw}} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \left(\frac{\nabla_{\mathbf{r}} n_\sigma}{n_\sigma} + \left(\frac{v^2}{2T_\sigma} - \frac{5}{2} \right) \frac{\nabla_{\mathbf{r}} T_\sigma}{T_\sigma} \right) \frac{\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}}{T_\sigma} \\
& + \frac{1}{2} [H_{01}^2]^{\text{lw}} \frac{\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}}{T_\sigma^2} - H_{02}^{\text{lw}} \frac{\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}}{T_\sigma}, \tag{G.1}
\end{aligned}$$

where the functions \mathbf{R}_{02} , H_{01} and H_{02} are given by

$$\mathbf{R} = \mathbf{r} + \frac{\epsilon_\sigma}{B} \mathbf{v} \times \hat{\mathbf{b}} + \epsilon_\sigma^2 \mathbf{R}_{02} + O(\epsilon_\sigma^3) \tag{G.2}$$

and

$$\frac{u^2}{2} + \mu B(\mathbf{R}) = \frac{v^2}{2} + \epsilon_\sigma H_{01} + \epsilon_\sigma^2 H_{02} + O(\epsilon_\sigma^3). \tag{G.3}$$

In what follows we calculate \mathbf{R}_{02} , H_{01} and H_{02} .

To compute \mathbf{R}_{02} we use that

$$\mathbf{r} = \mathbf{R}_g + \epsilon_\sigma \boldsymbol{\rho}(\mathbf{R}_g, \mu_g, \theta_g). \tag{G.4}$$

Employing the results in (C.4) it is easy to see that

$$\begin{aligned}
\mathbf{r} = \mathbf{R}_g - \frac{\epsilon_\sigma}{B} \mathbf{v} \times \hat{\mathbf{b}} + \epsilon_\sigma^2 \mathcal{T}_{\sigma,0}^{-1*} \left[-\boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \boldsymbol{\rho} \right. \\
+ \left(\frac{\mu}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B + u(\boldsymbol{\rho} \times \hat{\mathbf{b}}) \boldsymbol{\rho} : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \right) \partial_\mu \boldsymbol{\rho} \\
\left. - \left(\boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 - \frac{u}{2\mu} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} \right) \partial_\theta \boldsymbol{\rho} \right] + O(\epsilon_\sigma^3). \tag{G.5}
\end{aligned}$$

Using $\nabla_{\mathbf{R}} \boldsymbol{\rho} = -(2B)^{-1} \nabla_{\mathbf{R}} B \boldsymbol{\rho} - (\nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho}) \hat{\mathbf{b}} + (\nabla_{\mathbf{R}} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1) \boldsymbol{\rho} \times \hat{\mathbf{b}}$, $\partial_\mu \boldsymbol{\rho} = (2\mu)^{-1} \boldsymbol{\rho}$ and $\partial_\theta \boldsymbol{\rho} = -\boldsymbol{\rho} \times \hat{\mathbf{b}}$, we obtain

$$\begin{aligned}
\mathbf{r} = \mathbf{R}_g - \frac{\epsilon_\sigma}{B} \mathbf{v} \times \hat{\mathbf{b}} + \frac{\epsilon_\sigma^2}{B^2} \left[\frac{1}{B} (\mathbf{v} \times \hat{\mathbf{b}}) (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} B \right. \\
+ (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}}) \hat{\mathbf{b}} + v_{\parallel} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \times \hat{\mathbf{b}} \left. \right] \\
+ O(\epsilon_\sigma^3). \tag{G.6}
\end{aligned}$$

Finally, since $\mathbf{R}_g = \mathbf{R} + \epsilon_\sigma^2 \mathbf{R}_2 + O(\epsilon_\sigma^3)$ with \mathbf{R}_2 given in (67), we obtain

$$\begin{aligned}
\mathbf{R}_{02} = \frac{1}{B} \left[\left(v_{\parallel} \hat{\mathbf{b}} + \frac{1}{4} \mathbf{v}_{\perp} \right) \mathbf{v} \times \hat{\mathbf{b}} \right. \\
+ \mathbf{v} \times \hat{\mathbf{b}} \left(v_{\parallel} \hat{\mathbf{b}} + \frac{1}{4} \mathbf{v}_{\perp} \right) \left. \right] \cdot \nabla_{\mathbf{r}} \left(\frac{\hat{\mathbf{b}}}{B} \right) \\
+ \frac{v_{\parallel}}{B^2} \mathbf{v}_{\perp} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} + \frac{v_{\parallel}}{B^2} \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \mathbf{v}_{\perp}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\hat{\mathbf{b}}}{8B^2} [\mathbf{v}_\perp \mathbf{v}_\perp - (\mathbf{v}_\perp \times \hat{\mathbf{b}})(\mathbf{v}_\perp \times \hat{\mathbf{b}})] : \nabla_{\mathbf{r}} \hat{\mathbf{b}} \\
& + \frac{Z_\sigma \lambda_\sigma}{B^2} \hat{\mathbf{b}} \times \mathbb{T}_{\sigma,0} \nabla_{(\mathbf{R}_\perp/\epsilon_\sigma)} \tilde{\Phi}_\sigma \\
& + \frac{v_\perp^2}{2B^3} \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} B - \frac{v_\perp^2}{4B^3} \nabla_{\mathbf{r}_\perp} B,
\end{aligned} \tag{G.7}$$

where $\mathbf{a} \hat{\times} \vec{\mathbf{M}} = \mathbf{a} \times (\mathbf{b} \cdot \vec{\mathbf{M}})$. The long-wavelength component is

$$\begin{aligned}
\mathbf{R}_{02}^{\text{lw}} = & \frac{1}{B} \left[\left(v_\parallel \hat{\mathbf{b}} + \frac{1}{4} \mathbf{v}_\perp \right) \mathbf{v} \times \hat{\mathbf{b}} \right. \\
& + \left. \mathbf{v} \times \hat{\mathbf{b}} \left(v_\parallel \hat{\mathbf{b}} + \frac{1}{4} \mathbf{v}_\perp \right) \right] \hat{\times} \nabla_{\mathbf{r}} \left(\frac{\hat{\mathbf{b}}}{B} \right) \\
& + \frac{v_\parallel}{B^2} \mathbf{v}_\perp \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} + \frac{v_\parallel}{B^2} \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \mathbf{v}_\perp \\
& + \frac{\hat{\mathbf{b}}}{8B^2} [\mathbf{v}_\perp \mathbf{v}_\perp - (\mathbf{v}_\perp \times \hat{\mathbf{b}})(\mathbf{v}_\perp \times \hat{\mathbf{b}})] : \nabla_{\mathbf{r}} \hat{\mathbf{b}} \\
& + \frac{v_\perp^2}{2B^3} \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} B - \frac{v_\perp^2}{4B^3} \nabla_{\mathbf{r}_\perp} B.
\end{aligned} \tag{G.8}$$

To obtain H_{01} and H_{02}^{lw} , we use that the expressions of the Hamiltonian in the two different sets of variables are related (with some abuse of notation) by

$$\begin{aligned}
\frac{v^2}{2} + Z_\sigma \lambda_\sigma \epsilon_\sigma \varphi(\mathbf{r}, \mathbf{v}, t) = & \\
& \frac{u^2}{2} + \mu B(\mathbf{R}) + Z_\sigma \lambda_\sigma \epsilon_\sigma \langle \phi_\sigma \rangle(\mathbf{R}, \mu, t) \\
& + Z_\sigma^2 \lambda_\sigma^2 \epsilon_\sigma^2 \Psi_{\phi,\sigma} + Z_\sigma \lambda_\sigma \epsilon_\sigma^2 \Psi_{\phi B,\sigma} + \epsilon_\sigma^2 \Psi_{B,\sigma} \\
& - \frac{Z_\sigma \lambda_\sigma \epsilon_\sigma^2}{B} \partial_t \tilde{\Phi}_\sigma + O(\epsilon_\sigma^3).
\end{aligned} \tag{G.9}$$

Let us give a more detailed explanation of the last equation. As shown in reference [16], and to the order of interest, the Hamiltonian in gyrokinetic coordinates, \overline{H}_σ , is the Hamiltonian in cartesian coordinates, $H_\sigma^{\mathbf{X}}$, after a change of coordinates and the addition of the partial derivative with respect to time of a gauge function. This function is $-S_{NP,\sigma} - \epsilon_\sigma^2 S_{P,\sigma}^{(2)}$, where $S_{NP,\sigma}$ (which does not depend on time) and $S_{P,\sigma}^{(2)}$ are given in equations (81) and (108) of reference [16], respectively. As a result,

$$\overline{H}_\sigma = \mathcal{T}_\sigma^* H_\sigma^{\mathbf{X}} - \epsilon_\sigma^2 \partial_t S_{P,\sigma}^{(2)} + O(\epsilon_\sigma^3), \tag{G.10}$$

This is the origin of the last term in (G.9).

The function $\langle \phi_\sigma \rangle(\mathbf{R}, \mu, t)$ is

$$\begin{aligned}
\langle \phi_\sigma \rangle(\mathbf{R}, \mu, t) = & \langle \phi_\sigma \rangle(\mathbf{R}_g, \mu_g, t) \\
& - \epsilon_\sigma (\mathbf{R}_2 \cdot \nabla_{\mathbf{R}_{g\perp}/\epsilon_\sigma} \langle \phi_\sigma \rangle + \mu_1 \partial_{\mu_g} \langle \phi_\sigma \rangle) \\
& + O(\epsilon_\sigma^2).
\end{aligned} \tag{G.11}$$

Here it is worth distinguishing between long-wavelength and short-wavelength pieces. For the long-wavelength potential,

$$\begin{aligned} \langle \phi_\sigma^{\text{lw}} \rangle(\mathbf{R}_g, \mu_g, t) &= \frac{1}{\epsilon_\sigma \lambda_\sigma} \varphi_0(\mathbf{R}_g, t) + \varphi_1^{\text{lw}}(\mathbf{R}_g, t) \\ &+ \frac{\epsilon_\sigma \mu_g}{2 \lambda_\sigma B(\mathbf{R}_g)} \left(\overset{\leftrightarrow}{\mathbf{I}} - \hat{\mathbf{b}}(\mathbf{R}_g) \hat{\mathbf{b}}(\mathbf{R}_g) \right) : \nabla_{\mathbf{R}_g} \nabla_{\mathbf{R}_g} \varphi_0(\mathbf{R}_g, t) + O(\epsilon_\sigma^2). \end{aligned} \quad (\text{G.12})$$

This has to be written in (\mathbf{r}, \mathbf{v}) variables. Employing the results in (C.4) gives

$$\begin{aligned} \langle \phi_\sigma^{\text{lw}} \rangle(\mathbf{R}_g, \mu_g, t) &= \frac{1}{\epsilon_\sigma \lambda_\sigma} \varphi_0 + \varphi_1^{\text{lw}} + \frac{1}{\lambda_\sigma B} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \varphi_0 \\ &+ \frac{\epsilon_\sigma}{B} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \varphi_1^{\text{lw}} \\ &- \frac{1}{\lambda_\sigma} \mathcal{T}_{\sigma,0}^{-1*} \left[-\boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \boldsymbol{\rho} + \left(\frac{\mu}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B + u(\boldsymbol{\rho} \times \hat{\mathbf{b}}) \boldsymbol{\rho} : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \right) \partial_\mu \boldsymbol{\rho} \right. \\ &- \left. \left(\boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 - \frac{u}{2\mu} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} \right) \partial_\theta \boldsymbol{\rho} \right] \cdot \nabla_{\mathbf{r}} \varphi_0 \\ &+ \frac{\epsilon_\sigma}{2 \lambda_\sigma B^2} \left[(\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) + \frac{v_\perp^2}{2} (\overset{\leftrightarrow}{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \right] : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \varphi_0 + O(\epsilon_\sigma^2), \end{aligned} \quad (\text{G.13})$$

where on the right-hand side everything is evaluated at \mathbf{r} . With this result, we find that to lowest order

$$\begin{aligned} \frac{u^2}{2} + \mu B(\mathbf{R}) &= \frac{v^2}{2} - \frac{Z_\sigma \epsilon_\sigma}{B} (\mathbf{v} \times \hat{\mathbf{b}}(\mathbf{r})) \cdot \nabla_{\mathbf{r}} \varphi_0(\mathbf{r}, t) \\ &+ Z_\sigma \lambda_\sigma \epsilon_\sigma \mathbb{T}_{\sigma,0} \tilde{\phi}_{\sigma 1}^{\text{sw}}(\mathbf{r}, \mathbf{v}, t) + O(\epsilon_\sigma^2), \end{aligned} \quad (\text{G.14})$$

giving

$$H_{01}^{\text{lw}}(\mathbf{r}, \mathbf{v}, t) = -\frac{Z_\sigma}{B} (\mathbf{v} \times \hat{\mathbf{b}}(\mathbf{r})) \cdot \nabla_{\mathbf{r}} \varphi_0(\mathbf{r}, t) \quad (\text{G.15})$$

and

$$\begin{aligned} [H_{01}^2]^{\text{lw}}(\mathbf{r}, \mathbf{v}, t) &= \frac{Z_\sigma^2}{B^2} \left[(\mathbf{v} \times \hat{\mathbf{b}}(\mathbf{r})) \cdot \nabla_{\mathbf{r}} \varphi_0(\mathbf{r}, t) \right]^2 \\ &+ Z_\sigma^2 \lambda_\sigma^2 \mathcal{T}_{\sigma,0}^{-1*} \left[(\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right]^{\text{lw}}(\mathbf{r}, \mathbf{v}, t). \end{aligned} \quad (\text{G.16})$$

Going to higher order, we find

$$\begin{aligned} H_{02}^{\text{lw}}(\mathbf{r}, \mathbf{v}, t) &= -\frac{Z_\sigma \lambda_\sigma}{B} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \varphi_1^{\text{lw}} - Z_\sigma \mathbf{R}_{02}^{\text{lw}} \cdot \nabla_{\mathbf{r}} \varphi_0 \\ &- Z_\sigma^2 \lambda_\sigma^2 \mathcal{T}_{\sigma,0}^{-1*} \Psi_{\phi,\sigma}^{\text{lw}} - Z_\sigma \lambda_\sigma \mathcal{T}_{\sigma,0}^{-1*} \Psi_{\phi B,\sigma}^{\text{lw}} \\ &- \mathcal{T}_{\sigma,0}^{-1*} \Psi_{B,\sigma} - \frac{Z_\sigma}{2B^2} \left[(\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) + \frac{v_\perp^2}{2} (\overset{\leftrightarrow}{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \right] : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \varphi_0 \\ &+ \frac{Z_\sigma^2 \lambda_\sigma^2}{B} \mathcal{T}_{\sigma,0}^{-1*} \left[\left((\nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \tilde{\Phi}_\sigma^{\text{sw}}) \times \hat{\mathbf{b}} \right) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_\sigma^{\text{sw}} \rangle \right]^{\text{lw}} \\ &- \frac{Z_\sigma^2 \lambda_\sigma^2}{B} \mathcal{T}_{\sigma,0}^{-1*} \left[\tilde{\phi}_{\sigma 1}^{\text{sw}} \partial_\mu \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right]^{\text{lw}}. \end{aligned} \quad (\text{G.17})$$

Note that $[(\nabla_{\mathbf{R}_\perp/\epsilon_\sigma} \tilde{\Phi}_\sigma^{\text{sw}} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_\perp/\epsilon_\sigma} \langle \phi_\sigma^{\text{sw}} \rangle]^{\text{lw}} = O(\epsilon_\sigma)$ and can be neglected, giving

$$\begin{aligned}
H_{02}^{\text{lw}}(\mathbf{r}, \mathbf{v}, t) = & -\frac{Z_\sigma \lambda_\sigma}{B} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \varphi_1^{\text{lw}} - Z_\sigma \mathbf{R}_{02}^{\text{lw}} \cdot \nabla_{\mathbf{r}} \varphi_0 \\
& - Z_\sigma^2 \lambda_\sigma^2 \mathcal{T}_{\sigma,0}^{-1*} \Psi_{\phi,\sigma}^{\text{lw}} - Z_\sigma \lambda_\sigma \mathcal{T}_{\sigma,0}^{-1*} \Psi_{\phi B,\sigma}^{\text{lw}} - \mathcal{T}_{\sigma,0}^{-1*} \Psi_{B,\sigma} \\
& - \frac{Z_\sigma}{2B^2} \left[(\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) + \frac{v_\perp^2}{2} (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \right] : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \varphi_0 \\
& - \frac{Z_\sigma^2 \lambda_\sigma^2}{B} \mathcal{T}_{\sigma,0}^{-1*} \left[\tilde{\phi}_{\sigma 1}^{\text{sw}} \partial_\mu \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right]^{\text{lw}}. \tag{G.18}
\end{aligned}$$

Combining all these results we obtain

$$\begin{aligned}
[\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}} = & \frac{1}{2B^2} (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) : \left[\nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \ln n_\sigma + \frac{Z_\sigma}{T_\sigma} \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \varphi_0 \right. \\
& - \frac{Z_\sigma}{T_\sigma^2} (\nabla_{\mathbf{r}} \varphi_0 \nabla_{\mathbf{r}} T_\sigma + \nabla_{\mathbf{r}} T_\sigma \nabla_{\mathbf{r}} \varphi_0) + \left(\frac{v^2}{2T_\sigma} - \frac{3}{2} \right) \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \ln T_\sigma \\
& \left. - \frac{v^2}{2T_\sigma^3} \nabla_{\mathbf{r}} T_\sigma \nabla_{\mathbf{r}} T_\sigma \right] \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} + \frac{1}{2B^2} \left[(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \left(\frac{\nabla_{\mathbf{r}} n_\sigma}{n_\sigma} \right. \right. \\
& \left. \left. + \frac{Z_\sigma \nabla_{\mathbf{r}} \varphi_0}{T_\sigma} + \left(\frac{v^2}{2T_\sigma} - \frac{3}{2} \right) \frac{\nabla_{\mathbf{r}} T_\sigma}{T_\sigma} \right) \right]^2 \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \\
& + \mathbf{R}_{02}^{\text{lw}} \cdot \left(\frac{\nabla_{\mathbf{r}} n_\sigma}{n_\sigma} + \frac{Z_\sigma \nabla_{\mathbf{r}} \varphi_0}{T_\sigma} + \left(\frac{v^2}{2T_\sigma} - \frac{3}{2} \right) \frac{\nabla_{\mathbf{r}} T_\sigma}{T_\sigma} \right) \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0} \\
& + \frac{Z_\sigma^2 \lambda_\sigma^2}{2T_\sigma^2} \mathcal{T}_{\sigma,0}^{-1*} \left[(\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right]^{\text{lw}} F_{\sigma 0} + \frac{1}{T_\sigma} \left[\frac{Z_\sigma \lambda_\sigma}{B} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \varphi_1^{\text{lw}} \right. \\
& + Z_\sigma^2 \lambda_\sigma^2 \mathcal{T}_{\sigma,0}^{-1*} \Psi_{\phi,\sigma}^{\text{lw}} + Z_\sigma \lambda_\sigma \mathcal{T}_{\sigma,0}^{-1*} \Psi_{\phi B,\sigma}^{\text{lw}} \\
& + \mathcal{T}_{\sigma,0}^{-1*} \Psi_{B,\sigma} + \frac{Z_\sigma v_\perp^2}{4B^2} (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \varphi_0 \\
& \left. + \frac{Z_\sigma^2 \lambda_\sigma^2}{B} \mathcal{T}_{\sigma,0}^{-1*} \left[\tilde{\phi}_{\sigma 1}^{\text{sw}} \partial_\mu \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right]^{\text{lw}} \right] \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}. \tag{G.19}
\end{aligned}$$

Appendix H. Calculations for the Fokker-Planck equation to $O(\epsilon_\sigma^2)$

Start with (111). Employing the definition of $G_{\sigma 1}$ in (106), $\partial_\zeta F_{\sigma 1}^{\text{lw}} \equiv 0$, and using the identities

$$(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \varphi_0) \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} = \partial_\psi \varphi_0 I \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}}, \tag{H.1}$$

$$\begin{aligned}
(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B) \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} = & \partial_\psi B I \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} \\
& - \partial_\psi G_{\sigma 1}^{\text{lw}} I \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B, \tag{H.2}
\end{aligned}$$

$$\begin{aligned}
[\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} = & (\nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} - (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} = \\
& \mathbf{B} \cdot \nabla_{\mathbf{R}} \left(\frac{I}{B} \partial_\psi G_{\sigma 1}^{\text{lw}} \right)
\end{aligned}$$

$$\begin{aligned}
& -\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\psi} \left(\frac{1}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} I \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} \right) \\
& -(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}},
\end{aligned} \tag{H.3}$$

$$\begin{aligned}
& \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \cdot (\mu \nabla_{\mathbf{R}} B + Z_{\sigma} \nabla_{\mathbf{R}_{\perp}} \varphi_0) = \\
& \nabla_{\mathbf{R}} \cdot \left(\mu \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B + Z_{\sigma} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}_{\perp}} \varphi_0 \right) \\
& -\mu (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \\
& = \mathbf{B} \cdot \nabla_{\mathbf{R}} \left[\frac{I}{B} (\mu \partial_{\psi} B + Z_{\sigma} \partial_{\psi} \varphi_0) \right] \\
& -\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\psi} \left(\frac{I}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \right) \\
& -\mu (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B,
\end{aligned} \tag{H.4}$$

$$\begin{aligned}
& Z_{\sigma} \lambda_{\sigma} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \partial_u \left(\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{T_{\sigma}} F_{\sigma 0} \right) = \\
& - \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left[\frac{1}{2} \left(\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{T_{\sigma}} \right)^2 F_{\sigma 0} \right],
\end{aligned} \tag{H.5}$$

$$\begin{aligned}
& \frac{1}{B} \left[-Z_{\sigma} \nabla_{\mathbf{R}} \varphi_0 \times \hat{\mathbf{b}} + \mu \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B \right. \\
& \left. + u^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}} \left(\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{T_{\sigma}} F_{\sigma 0} \right) \\
& - Z_{\sigma} \lambda_{\sigma} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \partial_u \left[\frac{I u}{B} \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} \right) F_{\sigma 0} \right] \\
& - \frac{u}{B} \left[\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot (\mu \nabla_{\mathbf{R}} B \\
& + Z_{\sigma} \nabla_{\mathbf{R}} \varphi_0) \partial_u \left(\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{T_{\sigma}} F_{\sigma 0} \right) = \\
& - \frac{Z_{\sigma} \lambda_{\sigma} I}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \Upsilon_{\sigma} F_{\sigma 0} \\
& + \frac{Z_{\sigma} \lambda_{\sigma}}{B T_{\sigma}} F_{\sigma 0} [\mu \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B + u^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \\
& + \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left[\frac{Z_{\sigma} \lambda_{\sigma} I u}{T_{\sigma} B} \varphi_1^{\text{lw}} F_{\sigma 0} \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 \right. \right. \\
& \left. \left. + \Upsilon_{\sigma} - \frac{\partial_{\psi} T_{\sigma}}{T_{\sigma}} \right) \right],
\end{aligned} \tag{H.6}$$

$$\begin{aligned}
& \left[-\frac{Z_{\sigma}}{B} \nabla_{\mathbf{R}} \varphi_0 \times \hat{\mathbf{b}} + \frac{\mu}{B} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B \right. \\
& \left. + \frac{u^2}{B} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}} \left(\frac{I u}{B} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{u}{B}[\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot (\mu \nabla_{\mathbf{R}} B + Z_{\sigma} \nabla_{\mathbf{R}} \varphi_0) \partial_u \left(\frac{Iu}{B} \right) \\
& = -\frac{u}{B}(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \left[u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \left(\frac{Iu}{B} \right) - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \left(\frac{Iu}{B} \right) \right], \tag{H.7}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{B} \left[-Z_{\sigma} \nabla_{\mathbf{R}} \varphi_0 \times \hat{\mathbf{b}} + \mu \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B \right. \\
& \quad \left. + u^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}} \left(\left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} \right) F_{\sigma 0} \right) \\
& - \frac{u}{B} \left[\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot (\mu \nabla_{\mathbf{R}} B \\
& + Z_{\sigma} \nabla_{\mathbf{R}} \varphi_0) \partial_u \left(\left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} \right) F_{\sigma 0} \right) = \\
& \left[u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \left(\frac{Iu}{B} \right) - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \left(\frac{Iu}{B} \right) \right] \times \\
& \left[\partial_{\psi} \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 \right) - \frac{Z_{\sigma}}{T_{\sigma}^2} \partial_{\psi} \varphi_0 \partial_{\psi} T_{\sigma} + \partial_{\psi} \Upsilon_{\sigma} \right. \\
& \quad \left. - \frac{\mu}{T_{\sigma}^2} \partial_{\psi} B \partial_{\psi} T_{\sigma} + \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} \right)^2 \right] F_{\sigma 0}, \tag{H.8}
\end{aligned}$$

we obtain

$$\begin{aligned}
& \frac{1}{B} \left[-Z_{\sigma} \nabla_{\mathbf{R}} \varphi_0 \times \hat{\mathbf{b}} + \mu \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B \right. \\
& \quad \left. + u^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}} F_{\sigma 1}^{\text{lw}} \\
& - \left\{ Z_{\sigma} \lambda_{\sigma} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} + \frac{u}{B} [\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \right. \\
& \quad \left. \cdot (\mu \nabla_{\mathbf{R}} B + Z_{\sigma} \nabla_{\mathbf{R}} \varphi_0) \right\} \partial_u F_{\sigma 1}^{\text{lw}} \\
& = \frac{1}{B} (Z_{\sigma} \partial_{\psi} \varphi_0 + \mu \partial_{\psi} B) I \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} \\
& - \frac{I\mu}{B} \partial_{\psi} G_{\sigma 1}^{\text{lw}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \\
& - \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\psi} \left(\frac{Iu^2}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} \right) \\
& + u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \left(\frac{Iu}{B} \partial_{\psi} G_{\sigma 1}^{\text{lw}} \right) \\
& - Z_{\sigma} \lambda_{\sigma} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \partial_u G_{\sigma 1}^{\text{lw}} \\
& - u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \left[\frac{I}{B} (\mu \partial_{\psi} B + Z_{\sigma} \partial_{\psi} \varphi_0) \right] \partial_u G_{\sigma 1}^{\text{lw}} \\
& + \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\psi} \left(\frac{Iu\mu}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \right) \partial_u G_{\sigma 1}^{\text{lw}}
\end{aligned}$$

$$\begin{aligned}
& -\frac{u}{B}(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}) \left(u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left\{ G_{\sigma 1}^{\text{lw}} \right. \\
& -\frac{Iu}{B} F_{\sigma 0} \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} \right) \left. \right\} \\
& + \frac{Z_{\sigma} \lambda_{\sigma} I}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} F_{\sigma 0} \Upsilon_{\sigma} \\
& - \frac{Z_{\sigma} \lambda_{\sigma}}{T_{\sigma} B} F_{\sigma 0} \left[\mu\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B + u^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \\
& - \left(u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left\{ \frac{1}{2} \left(\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{T_{\sigma}} \right)^2 F_{\sigma 0} \right. \\
& + \frac{Z_{\sigma} \lambda_{\sigma} Iu}{T_{\sigma} B} \varphi_1^{\text{lw}} F_{\sigma 0} \left[\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} - \frac{1}{T_{\sigma}} \partial_{\psi} T_{\sigma} \right] \\
& + \frac{1}{2} \left(\frac{Iu}{B} \right)^2 F_{\sigma 0} \left[\partial_{\psi} \Upsilon_{\sigma} - \frac{\mu \partial_{\psi} B}{T_{\sigma}} \partial_{\psi} \ln T_{\sigma} \right. \\
& + \partial_{\psi} \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 \right) - \frac{Z_{\sigma}}{T_{\sigma}^2} \partial_{\psi} \varphi_0 \partial_{\psi} T_{\sigma} \\
& \left. \left. + \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} \right)^2 \right] \right\}. \tag{H.9}
\end{aligned}$$

To simplify this expression we use

$$\begin{aligned}
& \frac{1}{B} (Z_{\sigma} \partial_{\psi} \varphi_0 + \mu \partial_{\psi} B) I \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} = \\
& \partial_u \left[\frac{I}{B} (Z_{\sigma} \partial_{\psi} \varphi_0 + \mu \partial_{\psi} B) u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} \right] \\
& - \frac{Iu}{B} (Z_{\sigma} \partial_{\psi} \varphi_0 + \mu \partial_{\psi} B) \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} (\partial_u G_{\sigma 1}^{\text{lw}}), \tag{H.10}
\end{aligned}$$

$$\begin{aligned}
& u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \left(\frac{Iu}{B} \partial_{\psi} G_{\sigma 1}^{\text{lw}} \right) - \frac{I}{B} \partial_{\psi} G_{\sigma 1}^{\text{lw}} \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B = \\
& \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left(\frac{Iu}{B} \partial_{\psi} G_{\sigma 1}^{\text{lw}} \right) \\
& + \frac{Iu}{B} \partial_{\psi} \partial_{\mu} G_{\sigma 1}^{\text{lw}} \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B, \tag{H.11}
\end{aligned}$$

$$\begin{aligned}
& - Z_{\sigma} \lambda_{\sigma} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \partial_u G_{\sigma 1}^{\text{lw}} = \\
& - \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left(\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{u} \partial_u G_{\sigma 1}^{\text{lw}} \right) \\
& + \partial_u \left[\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{u} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} G_{\sigma 1}^{\text{lw}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u G_{\sigma 1}^{\text{lw}} \right) \right], \tag{H.12}
\end{aligned}$$

$$\begin{aligned}
& - u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \left[\frac{I}{B} (\mu \partial_{\psi} B + Z_{\sigma} \partial_{\psi} \varphi_0) \right] \partial_u G_{\sigma 1}^{\text{lw}} = \\
& - \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left[\frac{I}{B} (\mu \partial_{\psi} B + Z_{\sigma} \partial_{\psi} \varphi_0) \partial_u G_{\sigma 1}^{\text{lw}} \right]
\end{aligned}$$

$$\begin{aligned}
& -\partial_u \left[\frac{I}{B} (\mu \partial_\psi B + Z_\sigma \partial_\psi \varphi_0) \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u G_{\sigma 1}^{\text{lw}} \right] \\
& + \frac{Iu}{B} (\mu \partial_\psi B + Z_\sigma \partial_\psi \varphi_0) \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} (\partial_u G_{\sigma 1}^{\text{lw}}),
\end{aligned} \tag{H.13}$$

$$\begin{aligned}
& \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta \partial_\psi \left(\frac{Iu\mu}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \right) \partial_u G_{\sigma 1}^{\text{lw}} = \\
& \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta \partial_\psi \left(\frac{Iu\mu}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u G_{\sigma 1}^{\text{lw}} \right) \\
& - \frac{Iu\mu}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_\psi \partial_u G_{\sigma 1}^{\text{lw}},
\end{aligned} \tag{H.14}$$

and

$$\begin{aligned}
& \frac{Z_\sigma \lambda_\sigma I}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} F_{\sigma 0} \Upsilon_\sigma \\
& - \frac{Z_\sigma \lambda_\sigma}{T_\sigma B} F_{\sigma 0} \left[\mu \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B + u^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \\
& = \frac{Z_\sigma \lambda_\sigma}{B} (\nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} F_{\sigma 0} \\
& + \frac{Z_\sigma \lambda_\sigma u}{B} [\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \partial_u F_{\sigma 0}.
\end{aligned} \tag{H.15}$$

Employing these results in (H.9) gives

$$\begin{aligned}
& \frac{1}{B} \left[-Z_\sigma \nabla_{\mathbf{R}} \varphi_0 \times \hat{\mathbf{b}} + \mu \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B \right. \\
& \left. + u^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}} F_{\sigma 1}^{\text{lw}} \\
& - \left\{ Z_\sigma \lambda_\sigma \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} + \frac{u}{B} [\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})] \right. \\
& \left. \cdot (\mu \nabla_{\mathbf{R}} B + Z_\sigma \nabla_{\mathbf{R}} \varphi_0) \right\} \partial_u F_{\sigma 1}^{\text{lw}} \\
& = -\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta \partial_\psi \left[\frac{Iu}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma 1}^{\text{lw}} \right] \\
& + \partial_u \left\{ \left[\frac{I}{B} (Z_\sigma \partial_\psi \varphi_0 + \mu \partial_\psi B) + \frac{Z_\sigma \lambda_\sigma}{u} \varphi_1^{\text{lw}} \right] \right. \\
& \left. \times \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma 1}^{\text{lw}} \right\} \\
& - \frac{u}{B} \left(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right) \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left[G_{\sigma 1}^{\text{lw}} \right. \\
& \left. - \frac{Iu}{B} \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right) F_{\sigma 0} \right] \\
& + \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left\{ \frac{Iu}{B} \partial_\psi G_{\sigma 1}^{\text{lw}} - \frac{Z_\sigma \lambda_\sigma}{u} \varphi_1^{\text{lw}} \partial_u G_{\sigma 1}^{\text{lw}} \right. \\
& \left. - \frac{I}{B} (Z_\sigma \partial_\psi \varphi_0 + \mu \partial_\psi B) \partial_u G_{\sigma 1}^{\text{lw}} - \frac{1}{2} \left(\frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{T_\sigma} \right)^2 F_{\sigma 0} \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{Z_\sigma \lambda_\sigma I u}{T_\sigma B} \varphi_1^{\text{lw}} F_{\sigma 0} \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma - \partial_\psi \ln T_\sigma \right) \\
& -\frac{1}{2} \left(\frac{I u}{B} \right)^2 \left[\partial_\psi \Upsilon_\sigma - \frac{\mu \partial_\psi B}{T_\sigma} \partial_\psi \ln T_\sigma + \partial_\psi \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 \right) \right. \\
& \left. - \frac{Z_\sigma}{T_\sigma^2} \partial_\psi \varphi_0 \partial_\psi T_\sigma + \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right)^2 \right] \Big\} \\
& + \frac{Z_\sigma \lambda_\sigma}{B} \left(\nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \times \hat{\mathbf{b}} \right) \cdot \nabla_{\mathbf{R}} F_{\sigma 0} \\
& + \frac{Z_\sigma \lambda_\sigma u}{B} \left[\hat{\mathbf{b}} \times \left(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \right) \right] \cdot \nabla_{\mathbf{R}} \varphi_1^{\text{lw}} \partial_u F_{\sigma 0}.
\end{aligned} \tag{H.16}$$

We also manipulate the terms containing \mathbf{K} (defined by equation (57)) in (111). First, note that

$$\begin{aligned}
& -\frac{u\mu}{B} (\nabla_{\mathbf{R}} \times \mathbf{K})_\perp \cdot \nabla_{\mathbf{R}} F_{\sigma 0} \\
& + \frac{\mu}{B} (\nabla_{\mathbf{R}} \times \mathbf{K})_\perp \cdot (\mu \nabla_{\mathbf{R}} B + Z_\sigma \nabla_{\mathbf{R}} \varphi_0) \partial_u F_{\sigma 0} = \\
& -\frac{u\mu}{B} \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right) (\nabla_{\mathbf{R}} \times \mathbf{K}) \cdot \nabla_{\mathbf{R}} \psi F_{\sigma 0}.
\end{aligned} \tag{H.17}$$

Using Appendix I one finds

$$\begin{aligned}
(\nabla_{\mathbf{R}} \times \mathbf{K}) \cdot \nabla_{\mathbf{R}} \psi &= \mathbf{B} \cdot \nabla_{\mathbf{R}} \left(\frac{I}{2B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right. \\
& \left. + \frac{R}{|\nabla_{\mathbf{R}} \psi|^2} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) \right),
\end{aligned} \tag{H.18}$$

so

$$\begin{aligned}
& -\frac{u\mu}{B} (\nabla_{\mathbf{R}} \times \mathbf{K})_\perp \cdot \nabla_{\mathbf{R}} F_{\sigma 0} \\
& + \frac{\mu}{B} (\nabla_{\mathbf{R}} \times \mathbf{K})_\perp \cdot (\mu \nabla_{\mathbf{R}} B + Z_\sigma \nabla_{\mathbf{R}} \varphi_0) \partial_u F_{\sigma 0} = \\
& - \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left\{ \mu F_{\sigma 0} \right. \\
& \times \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right) \left(\frac{I}{2B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right. \\
& \left. \left. + \frac{R}{|\nabla_{\mathbf{R}} \psi|^2} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) \right) \right\}.
\end{aligned} \tag{H.19}$$

Hence, employing (H.16) and (H.19), and reorganizing, the second-order Fokker-Planck equation, (111), becomes

$$\begin{aligned}
& -B \partial_\theta F_{\sigma 3}^{\text{lw}} + \frac{\lambda_\sigma^2}{\tau_\sigma} \partial_{\epsilon_s^2 t} F_{\sigma 0} + \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) \left\{ F_{\sigma 2}^{\text{lw}} \right. \\
& + \left[\frac{Z_\sigma \lambda_\sigma^2}{T_\sigma} \varphi_2^{\text{lw}} + \frac{Z_\sigma}{T_\sigma} \frac{\mu}{2B} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \varphi_0 \right] F_{\sigma 0} \\
& \left. + \frac{1}{T_\sigma} (\Psi_{B,\sigma} + Z_\sigma \lambda_\sigma \Psi_{\phi B,\sigma}^{\text{lw}} + Z_\sigma^2 \lambda_\sigma^2 \Psi_{\phi,\sigma}^{\text{lw}}) F_{\sigma 0} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{Iu}{B} \partial_\psi G_{\sigma 1}^{\text{lw}} - \frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{u} \partial_u G_{\sigma 1}^{\text{lw}} \\
& - \frac{I}{B} (\mu \partial_\psi B + Z_\sigma \partial_\psi \varphi_0) \partial_u G_{\sigma 1}^{\text{lw}} - \frac{1}{2} \left(\frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{T_\sigma} \right)^2 F_{\sigma 0} \\
& - \frac{Z_\sigma \lambda_\sigma Iu}{T_\sigma B} \varphi_1^{\text{lw}} F_{\sigma 0} \left[\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma - \frac{1}{T_\sigma} \partial_\psi T_\sigma \right] \\
& - \frac{1}{2} \left(\frac{Iu}{B} \right)^2 F_{\sigma 0} \left[\partial_\psi \Upsilon_\sigma - \frac{\mu \partial_\psi B}{T_\sigma} \partial_\psi \ln T_\sigma \right. \\
& + \partial_\psi \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 \right) - \frac{Z_\sigma}{T_\sigma^2} \partial_\psi \varphi_0 \partial_\psi T_\sigma \\
& + \left. \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right)^2 \right] - \mu F_{\sigma 0} \left(\frac{I}{2B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right. \\
& + \frac{R}{|\nabla_{\mathbf{R}} \psi|^2} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) \left. \right) \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right) \\
& + \left. \left[\frac{Z_\sigma \lambda_\sigma}{u \mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} F_{\sigma 1}^{\text{sw}} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle) \cdot \nabla_{\mathbf{R}} \Theta \right]^{\text{lw}} \right\} \\
& - \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta \partial_\psi \left[\frac{Iu}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma 1}^{\text{lw}} \right] \\
& + \partial_u \left[\frac{I}{B} (\mu \partial_\psi B + Z_\sigma \partial_\psi \varphi_0) \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma 1}^{\text{lw}} \right] \\
& + \partial_u \left[\frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{u} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma 1}^{\text{lw}} \right] \\
& - \frac{u}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma 1}^{\text{lw}} \\
& + \frac{Z_\sigma \lambda_\sigma}{B} \mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_\psi \left[\frac{F_{\sigma 1}^{\text{sw}}}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \left(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right) \cdot \nabla_{\mathbf{R}} \psi \right]^{\text{lw}} \\
& - Z_\sigma \lambda_\sigma \partial_u \left[\left(\frac{\mu}{u B} (\hat{\mathbf{b}} \times \partial_\Theta B \nabla_{\mathbf{R}} \Theta) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1} \rangle^{\text{sw}} \right. \right. \\
& + \frac{u}{B} \left[\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}}) \right] \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1} \rangle^{\text{sw}} \\
& + \left. \left. \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_{\sigma 1} \rangle^{\text{sw}} \right) F_{\sigma 1}^{\text{sw}} \right]^{\text{lw}} \\
& = \sum_{\sigma'} \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'}^{(2)\text{lw}} + \sum_{\sigma'} \left[\mathcal{T}_{\sigma,1}^* C_{\sigma\sigma'}^{(1)} \right]^{\text{lw}}. \tag{H.20}
\end{aligned}$$

Finally, defining

$$\begin{aligned}
G_{\sigma 2}^{\text{lw}} & = \langle F_{\sigma 2}^{\text{lw}} \rangle \\
& + \left[\frac{Z_\sigma \lambda_\sigma^2}{T_\sigma} \varphi_2^{\text{lw}} + \frac{Z_\sigma}{T_\sigma} \frac{\mu}{2B} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \varphi_0 \right] F_{\sigma 0} \\
& + \frac{1}{T_\sigma} (\Psi_B + Z_\sigma \lambda_\sigma \Psi_{\phi B, \sigma}^{\text{lw}} + Z_\sigma^2 \lambda_\sigma^2 \Psi_{\phi, \sigma}^{\text{lw}}) F_{\sigma 0}
\end{aligned}$$

$$\begin{aligned}
& + \frac{Iu}{B} \partial_\psi G_{\sigma 1}^{\text{lw}} - \frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{u} \partial_u G_{\sigma 1}^{\text{lw}} \\
& - \frac{I}{B} (\mu \partial_\psi B + Z_\sigma \partial_\psi \varphi_0) \partial_u G_{\sigma 1}^{\text{lw}} - \frac{1}{2} \left(\frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{T_\sigma} \right)^2 F_{\sigma 0} \\
& - \frac{Z_\sigma \lambda_\sigma Iu}{T_\sigma B} \varphi_1^{\text{lw}} F_{\sigma 0} \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma - \frac{1}{T_\sigma} \partial_\psi T_\sigma \right) \\
& - \frac{1}{2} \left(\frac{Iu}{B} \right)^2 F_{\sigma 0} \left[\partial_\psi \Upsilon_\sigma - \frac{\mu \partial_\psi B}{T_\sigma} \partial_\psi \ln T_\sigma \right. \\
& + \partial_\psi \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 \right) - \frac{Z_\sigma}{T_\sigma^2} \partial_\psi \varphi_0 \partial_\psi T_\sigma + \left. \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right)^2 \right] \\
& - \mu F_{\sigma 0} \left(\frac{I}{2B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right. \\
& + \frac{R}{|\nabla_{\mathbf{R}} \psi|^2} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) \left. \right) \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right) \\
& + \left[\frac{Z_\sigma \lambda_\sigma}{u \mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} F_{\sigma 1}^{\text{sw}} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle) \cdot \nabla_{\mathbf{R}} \Theta \right]^{\text{lw}} \\
& - \left\langle \frac{1}{u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \Theta \sum_{\sigma'} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle ; \tag{H.21}
\end{aligned}$$

using the results in Appendix C and

$$B^{-1} \nabla_{\mathbf{R}} \cdot (B \boldsymbol{\rho}) + \partial_u \hat{u}_1 + \partial_\mu \hat{\mu}_1 + \partial_\theta \hat{\theta}_1 = \frac{u}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \tag{H.22}$$

to write

$$\begin{aligned}
& \left\langle \left[\mathcal{T}_{\sigma, 1}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right]^{\text{lw}} \right\rangle = \\
& \partial_u \left(\left\langle u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \left\langle \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \right) \\
& + \partial_\mu \left\{ \frac{u \mu}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \left\langle \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \right. \\
& - \left\langle \left(\frac{Z_\sigma}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \varphi_0 + \frac{\mu}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B + \frac{u^2}{B} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \cdot \boldsymbol{\rho} \right) \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \left. \right\} \\
& - \frac{u}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \left\langle \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle + \frac{1}{B} \nabla_{\mathbf{R}} \cdot \left\langle B \boldsymbol{\rho} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \\
& + \left[\mathcal{T}_{\sigma, 1}^* C_{\sigma \sigma'}^{(1)\text{sw}} \right]^{\text{lw}} ; \tag{H.23}
\end{aligned}$$

and employing (105) and (117), we can write the gyroaveraged, long-wavelength second-order Fokker-Planck equation as in (119). However, for some purposes, mainly in connection with the long-wavelength gyrokinetic quasineutrality equation, it is useful to recast equation (H.21) in a different fashion. After some straightforward algebra one gets

$$G_{\sigma 2}^{\text{lw}} = \langle F_{\sigma 2}^{\text{lw}} \rangle + \frac{Z_\sigma \lambda_\sigma^2}{T_\sigma} \varphi_2^{\text{lw}} F_{\sigma 0} + \frac{Iu}{B} \partial_\psi G_{\sigma 1}^{\text{lw}} - \frac{Z_\sigma \lambda_\sigma \varphi_1^{\text{lw}}}{u} \partial_u G_{\sigma 1}^{\text{lw}}$$

$$\begin{aligned}
& -\frac{I}{B}(\mu\partial_\psi B + Z_\sigma\partial_\psi\varphi_0)\partial_u G_{\sigma 1}^{\text{lw}} - \frac{1}{2}\left(\frac{Z_\sigma\lambda_\sigma\varphi_1^{\text{lw}}}{T_\sigma}\right)^2 F_{\sigma 0} \\
& -\frac{Z_\sigma\lambda_\sigma Iu}{T_\sigma B}\varphi_1^{\text{lw}}F_{\sigma 0}\left[\frac{Z_\sigma}{T_\sigma}\partial_\psi\varphi_0 + \Upsilon_\sigma - \frac{1}{T_\sigma}\partial_\psi T_\sigma\right] \\
& -\frac{1}{2B^2}((Iu)^2 + \mu B|\nabla_{\mathbf{R}}\psi|^2)\left[-\frac{2Z_\sigma}{T_\sigma^2}\partial_\psi\varphi_0\partial_\psi T_\sigma\right. \\
& \left.-\frac{u^2/2 + \mu B}{T_\sigma}(\partial_\psi \ln T_\sigma)^2 + \left(\frac{Z_\sigma}{T_\sigma}\partial_\psi\varphi_0 + \Upsilon_\sigma\right)^2\right]F_{\sigma 0} \\
& -\frac{1}{2}\left(\frac{Iu}{B}\right)^2 F_{\sigma 0}\left[\partial_\psi^2 \ln n_\sigma + \frac{Z_\sigma}{T_\sigma}\partial_\psi^2\varphi_0\right. \\
& \left.+ \left(\frac{u^2/2 + \mu B}{T_\sigma} - \frac{3}{2}\right)\partial_\psi^2 \ln T_\sigma\right] \\
& +\mu\left(\frac{1}{2B^2}\nabla_{\mathbf{R}}B \cdot \nabla_{\mathbf{R}}\psi - \frac{I}{2B}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}\right. \\
& \left.-\frac{R}{|\nabla_{\mathbf{R}}\psi|^2}\hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}}\nabla_{\mathbf{R}}\psi \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}}\psi)\right)\left(\frac{Z_\sigma}{T_\sigma}\partial_\psi\varphi_0 + \Upsilon_\sigma\right)F_{\sigma 0} \\
& +\left[\frac{Z_\sigma\lambda_\sigma}{u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\Theta}F_{\sigma 1}^{\text{sw}}(\hat{\mathbf{b}} \times \nabla_{\mathbf{R}\perp}/\epsilon_\sigma\langle\phi_{\sigma 1}^{\text{sw}}\rangle) \cdot \nabla_{\mathbf{R}}\Theta\right]^{\text{lw}} \\
& -\left\langle\frac{1}{u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\Theta}\boldsymbol{\rho} \cdot \nabla_{\mathbf{R}}\Theta \sum_{\sigma'}\mathcal{T}_{\sigma,0}^*C_{\sigma\sigma'}^{(1)\text{lw}}\right\rangle \\
& -\frac{\mu}{2B}(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \left[\nabla_{\mathbf{R}}\nabla_{\mathbf{R}} \ln n_\sigma\right. \\
& \left.+ \left(\frac{u^2/2 + \mu B}{T_\sigma} - \frac{3}{2}\right)\nabla_{\mathbf{R}}\nabla_{\mathbf{R}} \ln T_\sigma\right. \\
& \left.+ \frac{Z_\sigma}{T_\sigma}\nabla_{\mathbf{R}}\nabla_{\mathbf{R}}\varphi_0\right]F_{\sigma 0} - \frac{Z_\sigma^2\lambda_\sigma^2}{2T_\sigma^2}\left[\left\langle(\tilde{\phi}_{\sigma 1}^{\text{sw}})^2\right\rangle\right]^{\text{lw}}F_{\sigma 0} \\
& +\mu\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}\left(\frac{u}{B}\partial_\mu - \partial_u\right)F_{\sigma 1}^{\text{lw}} \\
& +\left\langle\mathcal{T}_{\sigma,0}^*[\mathcal{T}_{\sigma,1}^{-1*}F_{\sigma 1}^{\text{lw}}]^{\text{lw}}\right\rangle + \left\langle\mathcal{T}_{\sigma,0}^*[\mathcal{T}_{\sigma,2}^{-1*}F_{\sigma 0}]^{\text{lw}}\right\rangle, \tag{H.24}
\end{aligned}$$

where $\left\langle\mathcal{T}_{\sigma,0}^*[\mathcal{T}_{\sigma,1}^{-1*}F_{\sigma 1}^{\text{lw}}]^{\text{lw}}\right\rangle$ is given in (F.5) and $\left\langle\mathcal{T}_{\sigma,0}^*[\mathcal{T}_{\sigma,2}^{-1*}F_{\sigma 0}]^{\text{lw}}\right\rangle$ is given in (F.6). A less obvious calculation transforms the previous equation into

$$\begin{aligned}
G_{\sigma 2}^{\text{lw}} &= \langle F_{\sigma 2} \rangle^{\text{lw}} + \frac{Z_\sigma\lambda_\sigma^2}{T_\sigma}\varphi_2^{\text{lw}}F_{\sigma 0} + \frac{Iu}{B}\partial_\psi G_{\sigma 1}^{\text{lw}} - \frac{Z_\sigma\lambda_\sigma\varphi_1^{\text{lw}}}{u}\partial_u G_{\sigma 1}^{\text{lw}} \\
& -\frac{I}{B}(\mu\partial_\psi B + Z_\sigma\partial_\psi\varphi_0)\partial_u G_{\sigma 1}^{\text{lw}} - \frac{1}{2}\left(\frac{Z_\sigma\lambda_\sigma\varphi_1^{\text{lw}}}{T_\sigma}\right)^2 F_{\sigma 0} \\
& -\frac{Z_\sigma\lambda_\sigma Iu}{T_\sigma B}\varphi_1^{\text{lw}}F_{\sigma 0}\left(\frac{Z_\sigma}{T_\sigma}\partial_\psi\varphi_0 + \Upsilon_\sigma - \frac{1}{T_\sigma}\partial_\psi T_\sigma\right) \\
& -\frac{1}{2B^2}((Iu)^2 + \mu B|\nabla_{\mathbf{R}}\psi|^2)\left[-\frac{2Z_\sigma}{T_\sigma^2}\partial_\psi\varphi_0\partial_\psi T_\sigma\right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{Z_\sigma}{T_\sigma} \partial_\psi \varphi_0 + \Upsilon_\sigma \right)^2 + \partial_\psi^2 \ln n_\sigma \\
& + \left(\frac{u^2/2 + \mu B}{T_\sigma} - \frac{3}{2} \right) \partial_\psi^2 \ln T_\sigma \\
& - \frac{u^2/2 + \mu B}{T_\sigma} (\partial_\psi \ln T_\sigma)^2 + \frac{Z_\sigma}{T_\sigma} \partial_\psi^2 \varphi_0 \Big] F_{\sigma 0} \\
& + \left[\frac{Z_\sigma \lambda_\sigma}{u \mathbf{b} \cdot \nabla_{\mathbf{R}} \Theta} F_{\sigma 1}^{\text{sw}} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle) \cdot \nabla_{\mathbf{R}} \Theta \right]^{\text{lw}} \\
& - \left\langle \frac{1}{u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \Theta} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \Theta \sum_{\sigma'} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \\
& - \frac{Z_\sigma^2 \lambda_\sigma^2}{2 T_\sigma^2} \left[\left\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right\rangle \right]^{\text{lw}} F_{\sigma 0} + \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \left(\frac{u}{B} \partial_\mu - \partial_u \right) G_{\sigma 1}^{\text{lw}} \\
& + \left\langle \mathcal{T}_{\sigma, 0}^* [\mathcal{T}_{\sigma, 1}^{-1*} F_{\sigma 1}^{\text{lw}}]^{\text{lw}} \right\rangle + \left\langle \mathcal{T}_{\sigma, 0}^* [\mathcal{T}_{\sigma, 2}^{-1*} F_{\sigma 0}]^{\text{lw}} \right\rangle.
\end{aligned} \tag{H.25}$$

To obtain (H.25) from (H.24) we used equation (106) and

$$\begin{aligned}
& \frac{1}{B} \nabla_{\mathbf{R}} B \cdot \nabla_{\mathbf{R}} \psi + I \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \\
& - \frac{2RB}{|\nabla_{\mathbf{R}} \psi|^2} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) \\
& - \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi : (\overset{\leftrightarrow}{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) = 0.
\end{aligned} \tag{H.26}$$

Let us prove this. First, we have that

$$\begin{aligned}
\nabla_{\mathbf{R}} B \cdot \nabla_{\mathbf{R}} \psi &= \frac{I}{R^2 B} \nabla_{\mathbf{R}} I \cdot \nabla_{\mathbf{R}} \psi \\
& + \frac{1}{R^2 B} \nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} \psi - \frac{B}{R} \nabla_{\mathbf{R}} R \cdot \nabla_{\mathbf{R}} \psi,
\end{aligned} \tag{H.27}$$

where we have employed that $B^2 = (I^2 + |\nabla_{\mathbf{R}} \psi|^2)/R^2$. Noting that $\partial_\zeta(\nabla_{\mathbf{R}} \psi) = (\nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} R) \hat{\boldsymbol{\zeta}}$ we derive the following identities

$$\begin{aligned}
\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} &= \frac{1}{B^2} \mathbf{B} \cdot (\nabla_{\mathbf{R}} \times \mathbf{B}) \\
&= \frac{1}{B^2} \mathbf{B} \cdot [\nabla_{\mathbf{R}} I \times \nabla_{\mathbf{R}} \zeta + \nabla_{\mathbf{R}} \cdot (\nabla_{\mathbf{R}} \psi \nabla_{\mathbf{R}} \zeta) - \nabla_{\mathbf{R}} \cdot (\nabla_{\mathbf{R}} \zeta \nabla_{\mathbf{R}} \psi)] \\
&= \frac{1}{B^2} \mathbf{B} \cdot \left[\nabla_{\mathbf{R}} I \times \nabla_{\mathbf{R}} \zeta + \left(\nabla_{\mathbf{R}}^2 \psi - \frac{2}{R} \nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} R \right) \nabla_{\mathbf{R}} \zeta \right] \\
&= -\frac{1}{R^2 B^2} \nabla_{\mathbf{R}} I \cdot \nabla_{\mathbf{R}} \psi + \frac{I}{R^2 B^2} \nabla_{\mathbf{R}}^2 \psi - \frac{2I}{R^3 B^2} \nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} R,
\end{aligned} \tag{H.28}$$

$$\begin{aligned}
\hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) &= -\frac{|\nabla_{\mathbf{R}} \psi|^2}{RB} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot \hat{\boldsymbol{\zeta}} \\
& + \frac{I}{B} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\nabla_{\mathbf{R}} \zeta \times \nabla_{\mathbf{R}} \psi) \\
& = -\frac{|\nabla_{\mathbf{R}} \psi|^2}{R^2 B} \nabla_{\mathbf{R}} R \cdot \nabla_{\mathbf{R}} \psi,
\end{aligned} \tag{H.29}$$

$$\begin{aligned}
\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot \hat{\mathbf{b}} &= \frac{I^2}{R^2 B^2} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot \hat{\boldsymbol{\zeta}} \\
&+ \frac{2I}{RB^2} \hat{\boldsymbol{\zeta}} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \\
&+ \frac{1}{B^2} (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \\
&= \frac{I^2}{R^3 B^2} \nabla_{\mathbf{R}} R \cdot \nabla_{\mathbf{R}} \psi \\
&+ \frac{1}{B^2} (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi),
\end{aligned} \tag{H.30}$$

and

$$\begin{aligned}
&(\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \\
&= -(\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \cdot \nabla_{\mathbf{R}} (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \cdot \nabla_{\mathbf{R}} \psi \\
&= -\nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \cdot (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \\
&+ \nabla_{\mathbf{R}} \psi \cdot \{(\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi) \times [\nabla_{\mathbf{R}} \times (\nabla_{\mathbf{R}} \boldsymbol{\zeta} \times \nabla_{\mathbf{R}} \psi)]\} \\
&= -\nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} \left(\frac{|\nabla_{\mathbf{R}} \psi|^2}{2R^2} \right) \\
&+ \frac{|\nabla_{\mathbf{R}} \psi|^2}{R^2} \left(\nabla_{\mathbf{R}}^2 \psi - \frac{2}{R} \nabla_{\mathbf{R}} R \cdot \nabla_{\mathbf{R}} \psi \right) \\
&= -\frac{1}{R^2} \nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot \nabla_{\mathbf{R}} \psi \\
&+ \frac{|\nabla_{\mathbf{R}} \psi|^2}{R^2} \nabla_{\mathbf{R}}^2 \psi - \frac{|\nabla_{\mathbf{R}} \psi|^2}{R^3} \nabla_{\mathbf{R}} R \cdot \nabla_{\mathbf{R}} \psi.
\end{aligned} \tag{H.31}$$

Using relations (H.27), (H.28), (H.29), (H.30), and (H.31), it is trivial to check that (H.26) is satisfied.

Appendix I. Proof of (H.18)

First,

$$\begin{aligned}
(\nabla_{\mathbf{R}} \times \mathbf{K}) \cdot \nabla_{\mathbf{R}} \psi &= \\
&\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\Theta} \left[\frac{1}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \mathbf{K} \cdot (\nabla_{\mathbf{R}} \psi \times \nabla_{\mathbf{R}} \Theta) \right] \\
&+ \mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\zeta} \left[\frac{1}{\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta} \mathbf{K} \cdot (\nabla_{\mathbf{R}} \psi \times \nabla_{\mathbf{R}} \boldsymbol{\zeta}) \right].
\end{aligned} \tag{I.1}$$

Employing that $\partial_{\zeta} \mathbf{R} = (\nabla_{\mathbf{R}} \psi \times \nabla_{\mathbf{R}} \Theta) / (\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta) = R \hat{\boldsymbol{\zeta}}$ and $\partial_{\Theta} \mathbf{R} = -(\nabla_{\mathbf{R}} \psi \times \nabla_{\mathbf{R}} \boldsymbol{\zeta}) / (\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta)$, we find

$$\begin{aligned}
(\nabla_{\mathbf{R}} \times \mathbf{K}) \cdot \nabla_{\mathbf{R}} \psi &= \\
&\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\Theta} (\partial_{\zeta} \mathbf{R} \cdot \mathbf{K}) - \mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\zeta} (\partial_{\Theta} \mathbf{R} \cdot \mathbf{K}) \\
&= \mathbf{B} \cdot \nabla_{\mathbf{R}} \left(\frac{I}{2B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right) - \mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\Theta} (\partial_{\zeta} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1) \\
&+ \mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta \partial_{\zeta} (\partial_{\Theta} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1) = \mathbf{B} \cdot \nabla_{\mathbf{R}} \left(\frac{I}{2B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right)
\end{aligned}$$

$$-\mathbf{B} \cdot \nabla_{\mathbf{R}} \Theta (\partial_{\zeta} \hat{\mathbf{e}}_2 \cdot \partial_{\Theta} \hat{\mathbf{e}}_1 - \partial_{\Theta} \hat{\mathbf{e}}_2 \cdot \partial_{\zeta} \hat{\mathbf{e}}_1). \quad (\text{I.2})$$

Now, with the help of the relations $\nabla_{\mathbf{R}} \hat{\mathbf{e}}_1 = \nabla_{\mathbf{R}} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} + \nabla_{\mathbf{R}} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2$ and $\nabla_{\mathbf{R}} \hat{\mathbf{e}}_2 = \nabla_{\mathbf{R}} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} + \nabla_{\mathbf{R}} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1$, one gets

$$\begin{aligned} \partial_{\zeta} \hat{\mathbf{e}}_2 \cdot \partial_{\Theta} \hat{\mathbf{e}}_1 - \partial_{\Theta} \hat{\mathbf{e}}_2 \cdot \partial_{\zeta} \hat{\mathbf{e}}_1 &= \\ &= \left(\partial_{\zeta} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{b}} \right) \left(\partial_{\Theta} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{b}} \right) - \left(\partial_{\Theta} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{b}} \right) \left(\partial_{\zeta} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{b}} \right) \\ &= \left(\partial_{\zeta} \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_2 \right) \left(\partial_{\Theta} \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_1 \right) - \left(\partial_{\Theta} \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_2 \right) \left(\partial_{\zeta} \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_1 \right) \\ &= \left(\partial_{\Theta} \hat{\mathbf{b}} \times \partial_{\zeta} \hat{\mathbf{b}} \right) \cdot \hat{\mathbf{b}}. \end{aligned} \quad (\text{I.3})$$

Since this quantity is independent of the choice of $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, we can use $\hat{\mathbf{e}}_1 = \nabla_{\mathbf{R}} \psi / |\nabla_{\mathbf{R}} \psi|$ and $\hat{\mathbf{e}}_2 = (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) / |\nabla_{\mathbf{R}} \psi|$ without loss of generality, giving

$$\begin{aligned} \partial_{\zeta} \hat{\mathbf{e}}_2 \cdot \partial_{\Theta} \hat{\mathbf{e}}_1 - \partial_{\Theta} \hat{\mathbf{e}}_2 \cdot \partial_{\zeta} \hat{\mathbf{e}}_1 &= \partial_{\zeta} \left(\frac{\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi}{|\nabla_{\mathbf{R}} \psi|} \right) \cdot \partial_{\Theta} \left(\frac{\nabla_{\mathbf{R}} \psi}{|\nabla_{\mathbf{R}} \psi|} \right) \\ &\quad - \partial_{\Theta} \left(\frac{\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi}{|\nabla_{\mathbf{R}} \psi|} \right) \cdot \partial_{\zeta} \left(\frac{\nabla_{\mathbf{R}} \psi}{|\nabla_{\mathbf{R}} \psi|} \right) \\ &= -\partial_{\Theta} \left(\frac{1}{|\nabla_{\mathbf{R}} \psi|^2} \frac{\partial \nabla_{\mathbf{R}} \psi}{\partial \zeta} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) \right). \end{aligned} \quad (\text{I.4})$$

Thus,

$$\begin{aligned} (\nabla_{\mathbf{R}} \times \mathbf{K}) \cdot \nabla_{\mathbf{R}} \psi &= \mathbf{B} \cdot \nabla_{\mathbf{R}} \left(\frac{I}{2B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}} \right. \\ &\quad \left. + \frac{R}{|\nabla_{\mathbf{R}} \psi|^2} \hat{\zeta} \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \psi \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \psi) \right). \end{aligned} \quad (\text{I.5})$$

Appendix J. Some computations related to the long-wavelength quasineutrality equation

Firstly, let us show that (124) can be rewritten as in (125). Employ the relation (recall (69) and (70))

$$\partial_{\mu} \mu_{\sigma,1} + \partial_{\theta} \theta_{\sigma,1} = \frac{1}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B, \quad (\text{J.1})$$

the identity

$$\langle \boldsymbol{\rho} \boldsymbol{\rho} \rangle = \frac{\mu}{B} (\overset{\leftrightarrow}{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}), \quad (\text{J.2})$$

and the long-wavelength limit of (67) and (69),

$$\begin{aligned} \mathbf{R}_{\sigma,2}^{\text{lw}} &= -\frac{2u}{B} \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot (\boldsymbol{\rho} \times \hat{\mathbf{b}}) - \frac{1}{8} \hat{\mathbf{b}} \left[\boldsymbol{\rho} \boldsymbol{\rho} - (\boldsymbol{\rho} \times \hat{\mathbf{b}})(\boldsymbol{\rho} \times \hat{\mathbf{b}}) \right] : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \\ &\quad - \frac{u}{B} \hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} - \frac{1}{2B} \boldsymbol{\rho} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} B + O(\epsilon_{\sigma}), \end{aligned} \quad (\text{J.3})$$

$$\mu_{\sigma,1}^{\text{lw}} = -\frac{u^2}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}} \cdot \boldsymbol{\rho} + \frac{u}{4} \left[\boldsymbol{\rho}(\boldsymbol{\rho} \times \hat{\mathbf{b}}) + (\boldsymbol{\rho} \times \hat{\mathbf{b}})\boldsymbol{\rho} \right] : \nabla_{\mathbf{R}} \hat{\mathbf{b}} \quad (\text{J.4})$$

$$- \frac{Z_{\sigma}}{B} \boldsymbol{\rho} \cdot \nabla_{\mathbf{R}} \varphi_0 + O(\epsilon_{\sigma}), \quad (\text{J.5})$$

to recast (124) into

$$\begin{aligned}
& \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}^2} \int (B F_{\sigma 2}^{\text{lw}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \mathbf{K} F_{\sigma 0} + u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \hat{\mathbf{b}} F_{\sigma 1}^{\text{lw}}) du d\mu d\theta \\
& + 2\pi \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}^2} \left[\nabla_{\mathbf{r}} \cdot \left(\frac{3}{2B} \nabla_{\mathbf{r}_{\perp}} B \int \mu F_{\sigma 0} du d\mu \right) \right. \\
& + \frac{1}{2} \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} : \left((\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \int \mu F_{\sigma 0} du d\mu \right) \\
& - \nabla_{\mathbf{r}} \cdot \left(\frac{1}{B} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \int u^2 \mu \partial_{\mu} F_{\sigma 0} du d\mu \right) \\
& \left. - \nabla_{\mathbf{r}} \cdot \left(\frac{Z_{\sigma}}{B} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{r}} \varphi_0 \int \mu \partial_{\mu} F_{\sigma 0} du d\mu \right) \right] = 0. \tag{J.6}
\end{aligned}$$

We can get more explicit expressions by noting that the integrals containing $F_{\sigma 0}$ can be worked out analytically. Namely, if

$$F_{\sigma 0} = \frac{n_{\sigma}}{(2\pi T_{\sigma})^{3/2}} \exp \left(-\frac{\mu B + u^2/2}{T_{\sigma}} \right), \tag{J.7}$$

then

$$\partial_{\mu} F_{\sigma 0} = -\frac{B}{T_{\sigma}} F_{\sigma 0}, \tag{J.8}$$

and

$$\int \mu F_{\sigma 0} du d\mu = \frac{n_{\sigma} T_{\sigma}}{2\pi B^2}, \quad \int u^2 \mu F_{\sigma 0} du d\mu = \frac{n_{\sigma} T_{\sigma}^2}{2\pi B^2}. \tag{J.9}$$

With these results, equation (J.6) finally becomes (125).

Now, we proceed to recast (125) into (127) by using the function $G_{\sigma 2}^{\text{lw}}$ defined in (H.25). A simple rewriting of (125) in terms of $G_{\sigma 2}^{\text{lw}}$ gives

$$\begin{aligned}
& \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}^2} \int B \left\{ G_{\sigma 2}^{\text{lw}} - \frac{Z_{\sigma} \lambda_{\sigma}^2}{T_{\sigma}} \varphi_2^{\text{lw}} F_{\sigma 0} - \frac{Iu}{B} \partial_{\psi} G_{\sigma 1}^{\text{lw}} \right. \\
& + \frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{u} \partial_u G_{\sigma 1}^{\text{lw}} + \frac{1}{2} \left(\frac{Z_{\sigma} \lambda_{\sigma} \varphi_1^{\text{lw}}}{T_{\sigma}} \right)^2 F_{\sigma 0} \\
& + \frac{1}{2B^2} ((Iu)^2 + \mu B |\nabla_{\mathbf{r}} \psi|^2) \left[-\frac{2Z_{\sigma}}{T_{\sigma}^2} \partial_{\psi} \varphi_0 \partial_{\psi} T_{\sigma} \right. \\
& + \left(\frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi} \varphi_0 + \Upsilon_{\sigma} \right)^2 + \partial_{\psi}^2 \ln n_{\sigma} \\
& + \left(\frac{u^2/2 + \mu B}{T_{\sigma}} - \frac{3}{2} \right) \partial_{\psi}^2 \ln T_{\sigma} \\
& \left. - \frac{u^2/2 + \mu B}{T_{\sigma}} (\partial_{\psi} \ln T_{\sigma})^2 + \frac{Z_{\sigma}}{T_{\sigma}} \partial_{\psi}^2 \varphi_0 \right] F_{\sigma 0}
\end{aligned}$$

$$\begin{aligned}
& - \left[\frac{Z_\sigma}{u \hat{\mathbf{B}} \cdot \nabla_{\mathbf{r}} \Theta} F_{\sigma 1}^{\text{sw}} (\hat{\mathbf{b}} \times \nabla_{\mathbf{r}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle) \cdot \nabla_{\mathbf{r}} \Theta \right]^{\text{lw}} \\
& + \left\langle \frac{1}{u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \Theta} \boldsymbol{\rho} \cdot \nabla_{\mathbf{r}} \Theta \sum_{\sigma'} \mathcal{T}_{\sigma, 0}^* C_{\sigma \sigma'}^{(1)\text{lw}} \right\rangle \\
& + \frac{Z_\sigma^2 \lambda_\sigma^2}{2 T_\sigma^2} \left[\left\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right\rangle \right]^{\text{lw}} F_{\sigma 0} \Big\} du d\mu d\theta \\
& + \sum_{\sigma} \frac{Z_\sigma}{\lambda_\sigma^2} \int u \hat{\mathbf{b}} \cdot (\nabla_{\mathbf{r}} \times \hat{\mathbf{b}}) G_{\sigma 1}^{\text{lw}} du d\mu d\theta = 0.
\end{aligned} \tag{J.10}$$

Here we have used (F.5) to write

$$\begin{aligned}
& \int B \mathcal{T}_{\sigma, 0}^* \left\langle [\mathcal{T}_{\sigma, 1}^{-1*} F_{\sigma 1}^{\text{lw}}]^{\text{lw}} \right\rangle du d\mu d\theta \\
& = \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \hat{\mathbf{b}} \int u F_{\sigma 1}^{\text{lw}} du d\mu d\theta,
\end{aligned} \tag{J.11}$$

and we have employed the result in Appendix K:

$$\begin{aligned}
& \int B \left\langle \mathcal{T}_{\sigma, 0}^* [\mathcal{T}_{\sigma, 2}^{-1*} F_{\sigma 0}]^{\text{lw}} \right\rangle du d\mu d\theta = \\
& \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} : \left[\frac{n_\sigma T_\sigma}{2 B^2} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \right] + \nabla_{\mathbf{r}} \cdot \left(\frac{Z_\sigma n_\sigma}{B^2} \nabla_{\mathbf{r}_\perp} \varphi_0 \right) \\
& + \nabla_{\mathbf{r}} \cdot \left(\frac{3 n_\sigma T_\sigma}{2 B^3} \nabla_{\mathbf{r}_\perp} B \right) + \nabla_{\mathbf{r}} \cdot \left(\frac{n_\sigma T_\sigma}{B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \right) \\
& - \frac{n_\sigma T_\sigma}{B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \mathbf{K}.
\end{aligned} \tag{J.12}$$

Equation (J.10) can be simplified even more employing

$$\begin{aligned}
& \int B ((Iu)^2 + \mu B |\nabla_{\mathbf{r}} \psi|^2) F_{\sigma 0} du d\mu d\theta \\
& = (RB)^2 n_\sigma T_\sigma \\
& \int B ((Iu)^2 + \mu B |\nabla_{\mathbf{r}} \psi|^2) \frac{u^2/2 + \mu B}{T_\sigma} F_{\sigma 0} du d\mu d\theta \\
& = \frac{5}{2} (RB)^2 n_\sigma T_\sigma \\
& \int B ((Iu)^2 + \mu B |\nabla_{\mathbf{r}} \psi|^2) \left(\frac{u^2/2 + \mu B}{T_\sigma} - \frac{3}{2} \right) F_{\sigma 0} du d\mu d\theta \\
& = (RB)^2 n_\sigma T_\sigma \\
& \int B ((Iu)^2 + \mu B |\nabla_{\mathbf{r}} \psi|^2) \left(\frac{u^2/2 + \mu B}{T_\sigma} - \frac{3}{2} \right)^2 F_{\sigma 0} du d\mu d\theta \\
& = \frac{7}{2} (RB)^2 n_\sigma T_\sigma,
\end{aligned} \tag{J.13}$$

finally giving (127).

Appendix K. Integral of the second-order piece of the transformation of the Maxwellian

In this Appendix we calculate the integral in velocity space (J.12). The integrand $\langle \mathcal{T}_{\sigma,0}^* [\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}} \rangle$ is given in (F.6). Using

$$\begin{aligned} \frac{1}{T_\sigma} \int B \Psi_{B,\sigma} F_{\sigma 0} \, d\mathbf{u} d\mu d\theta = & \\ & -\frac{3n_\sigma T_\sigma}{2B^3} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} B + \frac{n_\sigma T_\sigma}{2B^3} (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \mathbf{B} \cdot \hat{\mathbf{b}} \\ & -\frac{3n_\sigma T_\sigma}{2B^4} |\nabla_{\mathbf{r}_\perp} B|^2 + \frac{n_\sigma T_\sigma}{2B^2} \nabla_{\mathbf{r}} \hat{\mathbf{b}} : \nabla_{\mathbf{r}} \hat{\mathbf{b}} \\ & -\frac{n_\sigma T_\sigma}{2B^2} (\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}})^2 \end{aligned} \quad (\text{K.1})$$

and

$$\begin{aligned} \int B \left(\frac{Z_\sigma^2 \lambda_\sigma^2}{2T_\sigma^2} \left[\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \rangle \right]^{\text{lw}} \right. \\ \left. - \frac{Z_\sigma^2 \lambda_\sigma^2}{2BT_\sigma} \partial_\mu \left[\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \rangle \right]^{\text{lw}} \right) F_{\sigma 0} \, d\mathbf{u} d\mu d\theta = \\ -\frac{Z_\sigma^2 \lambda_\sigma^2}{2T_\sigma} \int \partial_\mu \left(\left[\langle (\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \rangle \right]^{\text{lw}} F_{\sigma 0} \right) \, d\mathbf{u} d\mu d\theta = 0, \end{aligned} \quad (\text{K.2})$$

we obtain

$$\begin{aligned} \int B \langle \mathcal{T}_{\sigma,0}^* [\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}} \rangle \, d\mathbf{u} d\mu d\theta \\ = \frac{1}{2B^2} (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} (n_\sigma T_\sigma) + \nabla_{\mathbf{r}} \cdot \left(\frac{Z_\sigma n_\sigma}{B^2} \nabla_{\mathbf{r}_\perp} \varphi_0 \right) \\ - \frac{1}{2B^3} \nabla_{\mathbf{r}_\perp} B \cdot \nabla_{\mathbf{r}_\perp} (n_\sigma T_\sigma) - \frac{3n_\sigma T_\sigma}{2B^3} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} B \\ + \frac{n_\sigma T_\sigma}{2B^3} (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \mathbf{B} \cdot \hat{\mathbf{b}} - \frac{3n_\sigma T_\sigma}{2B^4} |\nabla_{\mathbf{r}_\perp} B|^2 \\ + \frac{n_\sigma T_\sigma}{2B^2} \nabla_{\mathbf{r}} \hat{\mathbf{b}} : \nabla_{\mathbf{r}} \hat{\mathbf{b}} - \frac{n_\sigma T_\sigma}{2B^2} (\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}})^2. \end{aligned} \quad (\text{K.3})$$

Using

$$\begin{aligned} (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} \mathbf{B} \cdot \hat{\mathbf{b}} = \\ (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} B - B \nabla_{\mathbf{r}} \hat{\mathbf{b}} : (\nabla_{\mathbf{r}} \hat{\mathbf{b}})^{\text{T}} + B |\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}}|^2 \end{aligned} \quad (\text{K.4})$$

and

$$\begin{aligned} \frac{1}{2B^2} (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} (n_\sigma T_\sigma) = \\ \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} : \left[\frac{n_\sigma T_\sigma}{2B^2} (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \right] + \frac{2}{B^3} \nabla_{\mathbf{r}_\perp} B \cdot \nabla_{\mathbf{r}_\perp} (n_\sigma T_\sigma) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} (n_{\sigma} T_{\sigma}) + \frac{n_{\sigma} T_{\sigma}}{B^3} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} B \\
& + \frac{5n_{\sigma} T_{\sigma}}{2B^2} (\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}})^2 - \frac{2n_{\sigma} T_{\sigma}}{B^3} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} B \\
& + \frac{n_{\sigma} T_{\sigma}}{B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} (\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}}) + \frac{n_{\sigma} T_{\sigma}}{2B^2} \nabla_{\mathbf{r}} \hat{\mathbf{b}} : \nabla_{\mathbf{r}} \hat{\mathbf{b}} \\
& - \frac{3n_{\sigma} T_{\sigma}}{B^4} |\nabla_{\mathbf{r}\perp} B|^2,
\end{aligned} \tag{K.5}$$

equation (K.3) can be rewritten as

$$\begin{aligned}
& \int B \left\langle \mathcal{T}_{\sigma,0}^* [\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}} \right\rangle d\mathbf{u} d\mu d\theta = \\
& \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} : \left[\frac{n_{\sigma} T_{\sigma}}{2B^2} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \right] + \nabla_{\mathbf{r}} \cdot \left(\frac{Z_{\sigma} n_{\sigma}}{B^2} \nabla_{\mathbf{r}\perp} \varphi_0 \right) \\
& + \frac{3}{2B^3} \nabla_{\mathbf{r}\perp} B \cdot \nabla_{\mathbf{r}\perp} (n_{\sigma} T_{\sigma}) + \frac{1}{B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} (n_{\sigma} T_{\sigma}) \\
& - \frac{7n_{\sigma} T_{\sigma}}{2B^3} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} B + \frac{3n_{\sigma} T_{\sigma}}{2B^3} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} B \\
& - \frac{n_{\sigma} T_{\sigma}}{2B^2} \nabla_{\mathbf{r}} \hat{\mathbf{b}} : (\nabla_{\mathbf{r}} \hat{\mathbf{b}})^{\text{T}} + \frac{n_{\sigma} T_{\sigma}}{2B^2} |\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}}|^2 \\
& - \frac{9n_{\sigma} T_{\sigma}}{2B^4} |\nabla_{\mathbf{r}\perp} B|^2 + \frac{n_{\sigma} T_{\sigma}}{B^2} \nabla_{\mathbf{r}} \hat{\mathbf{b}} : \nabla_{\mathbf{r}} \hat{\mathbf{b}} + \frac{2n_{\sigma} T_{\sigma}}{B^2} (\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}})^2 \\
& + \frac{n_{\sigma} T_{\sigma}}{B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} (\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}}).
\end{aligned} \tag{K.6}$$

With further manipulations, we find

$$\begin{aligned}
& \int B \left\langle \mathcal{T}_{\sigma,0}^* [\mathcal{T}_{\sigma,2}^{-1*} F_{\sigma 0}]^{\text{lw}} \right\rangle d\mathbf{u} d\mu d\theta = \\
& \nabla_{\mathbf{r}} \nabla_{\mathbf{r}} : \left[\frac{n_{\sigma} T_{\sigma}}{2B^2} (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \right] + \nabla_{\mathbf{r}} \cdot \left(\frac{Z_{\sigma} n_{\sigma}}{B^2} \nabla_{\mathbf{r}\perp} \varphi_0 \right) \\
& + \nabla_{\mathbf{r}} \cdot \left(\frac{3n_{\sigma} T_{\sigma}}{2B^3} \nabla_{\mathbf{r}\perp} B \right) + \nabla_{\mathbf{r}} \cdot \left(\frac{n_{\sigma} T_{\sigma}}{B^2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}} \right) \\
& - \frac{n_{\sigma} T_{\sigma}}{2B^2} \nabla_{\mathbf{r}} \hat{\mathbf{b}} : (\nabla_{\mathbf{r}} \hat{\mathbf{b}})^{\text{T}} + \frac{n_{\sigma} T_{\sigma}}{2B^2} |\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}}|^2 \\
& + \frac{n_{\sigma} T_{\sigma}}{2B^2} (\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}})^2.
\end{aligned} \tag{K.7}$$

Finally, we show that we can combine the last three terms of the previous equation to give a more recognizable term. Employing

$$\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \mathbf{K} = \frac{1}{2} (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \hat{\mathbf{b}})^2 - (\hat{\mathbf{b}} \times \nabla_{\mathbf{r}} \hat{\mathbf{e}}_1) : (\nabla_{\mathbf{r}} \hat{\mathbf{e}}_2)^{\text{T}}, \tag{K.8}$$

$$\begin{aligned}
\nabla_{\mathbf{r}} \hat{\mathbf{e}}_1 \cdot (\nabla_{\mathbf{r}} \hat{\mathbf{e}}_2)^{\text{T}} &= (\nabla_{\mathbf{r}} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{b}}) (\nabla_{\mathbf{r}} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{b}}) = \\
& (\nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_1) (\nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_2) = \frac{1}{2} \nabla_{\mathbf{r}} \hat{\mathbf{b}} \cdot (\nabla_{\mathbf{r}} \hat{\mathbf{b}} \times \hat{\mathbf{b}})^{\text{T}},
\end{aligned} \tag{K.9}$$

and

$$\hat{\mathbf{b}} \times \nabla_{\mathbf{r}} \hat{\mathbf{b}} \times \hat{\mathbf{b}} = (\nabla_{\mathbf{r}} \hat{\mathbf{b}})^{\text{T}} - (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \hat{\mathbf{b}}) \hat{\mathbf{b}} - (\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}}) (\hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}}), \tag{K.10}$$

one finds the identity

$$\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \mathbf{K} = \frac{1}{2}(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \hat{\mathbf{b}})^2 + \frac{1}{2}\nabla_{\mathbf{r}}\hat{\mathbf{b}} : \nabla_{\mathbf{r}}\hat{\mathbf{b}} - \frac{1}{2}(\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}})^2. \quad (\text{K.11})$$

Since $\nabla_{\mathbf{r}}\hat{\mathbf{b}} : (\nabla_{\mathbf{r}}\hat{\mathbf{b}})^{\text{T}} - \nabla_{\mathbf{r}}\hat{\mathbf{b}} : \nabla_{\mathbf{r}}\hat{\mathbf{b}} = |\nabla_{\mathbf{r}} \times \hat{\mathbf{b}}|^2$ and $\nabla_{\mathbf{r}} \times \hat{\mathbf{b}} = \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}}\hat{\mathbf{b}})$, we obtain

$$\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}} \times \mathbf{K} = \frac{1}{2}\nabla_{\mathbf{r}}\hat{\mathbf{b}} : (\nabla_{\mathbf{r}}\hat{\mathbf{b}})^{\text{T}} - \frac{1}{2}|\hat{\mathbf{b}} \cdot \nabla_{\mathbf{r}}\hat{\mathbf{b}}|^2 - \frac{1}{2}(\nabla_{\mathbf{r}} \cdot \hat{\mathbf{b}})^2, \quad (\text{K.12})$$

giving equation (J.12). We point out that $\nabla_{\mathbf{r}} \times \mathbf{K}$ was computed for the first time by Littlejohn in reference [42].

Appendix L. Proof of (140)

In this Appendix we prove (140). To do so, we take the short-wavelength quasineutrality equation to first order, given in (110), apply the operator $\partial_t + B^{-1}(\hat{\mathbf{b}} \times \nabla_{\mathbf{r}}\varphi_0(\mathbf{r}, t)) \cdot \nabla_{\mathbf{r}_{\perp}/\epsilon_s}$, and multiply it by $\varphi_1^{\text{sw}}(\mathbf{r}, t)$ to find

$$\begin{aligned} & \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}} \varphi_1^{\text{sw}}(\mathbf{r}, t) \left(\partial_t + \frac{Z_{\sigma}\tau_{\sigma}}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{r}}\varphi_0(\mathbf{r}, t)) \cdot \nabla_{\mathbf{r}_{\perp}/\epsilon_{\sigma}} \right) \int B \left[\right. \\ & \quad \left. - Z_{\sigma}\lambda_{\sigma}\tilde{\phi}_{\sigma 1}^{\text{sw}}(\mathbf{r} - \epsilon_{\sigma}\boldsymbol{\rho}(\mathbf{r}, \mu, \theta), \mu, \theta, t) \frac{F_{\sigma 0}(\mathbf{r}, u, \mu, t)}{T_{\sigma}(\mathbf{r}, t)} \right. \\ & \quad \left. + F_{\sigma 1}^{\text{sw}}(\mathbf{r} - \epsilon_{\sigma}\boldsymbol{\rho}(\mathbf{r}, \mu, \theta), u, \mu, t) \right] du d\mu d\theta = 0. \end{aligned} \quad (\text{L.1})$$

Since

$$\varphi_1^{\text{sw}}(\mathbf{r}, t) = \phi_{\sigma 1}^{\text{sw}}(\mathbf{r} - \epsilon_{\sigma}\boldsymbol{\rho}(\mathbf{r}, \mu, \theta), \mu, \theta, t) + O(\epsilon_{\sigma}), \quad (\text{L.2})$$

$$\nabla_{\mathbf{r}}\varphi_0(\mathbf{r}, t) = \nabla_{\mathbf{R}}\varphi_0(\mathbf{R}, t) + O(\epsilon_{\sigma}) \quad (\text{L.3})$$

and

$$\nabla_{\mathbf{r}_{\perp}/\epsilon_{\sigma}} = \nabla_{\mathbf{R}_{\perp}/\epsilon_{\sigma}} + O(\epsilon_{\sigma}), \quad (\text{L.4})$$

we find that (L.1) becomes

$$\begin{aligned} & \sum_{\sigma} \frac{Z_{\sigma}}{\lambda_{\sigma}} \int B \left[\phi_{\sigma 1}^{\text{sw}} \left(\partial_t + \frac{Z_{\sigma}\tau_{\sigma}}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}}\varphi_0) \cdot \nabla_{\mathbf{R}_{\perp}/\epsilon_{\sigma}} \right) \right. \\ & \quad \left. \left(- \frac{Z_{\sigma}\lambda_{\sigma}\tilde{\phi}_{\sigma 1}^{\text{sw}}}{T_{\sigma}} F_{\sigma 0} + F_{\sigma 1}^{\text{sw}} \right) \right]^{\text{lw}} du d\mu d\theta = O(\epsilon_s). \end{aligned} \quad (\text{L.5})$$

In this expression, the functions $\phi_{\sigma 1}^{\text{sw}}$ and $F_{\sigma 1}^{\text{sw}}$ are evaluated at $\mathbf{R} = \mathbf{r} - \epsilon_{\sigma}\boldsymbol{\rho}(\mathbf{r}, \mu, \theta)$, but after the coarse-grain average we can Taylor expand and, to lowest order, they can be evaluated at $\mathbf{R} = \mathbf{r}$. Thus, we find

$$- \sum_{\sigma} Z_{\sigma}^2 \int \frac{BF_{\sigma 0}}{2T_{\sigma}} \left(\partial_t \left[(\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right] \right)^{\text{lw}}$$

$$\begin{aligned}
& + \frac{Z_\sigma \tau_\sigma}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \varphi_0) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \left[(\tilde{\phi}_{\sigma 1}^{\text{sw}})^2 \right]^{\text{lw}} \Big) \text{d}u \text{d}\mu \text{d}\theta \\
& + \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma} \int B \left[\langle \phi_{\sigma 1}^{\text{sw}} \rangle \left(\partial_t F_{\sigma 1}^{\text{sw}} \right. \right. \\
& \left. \left. + \frac{Z_\sigma \tau_\sigma}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \varphi_0) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} F_{\sigma 1}^{\text{sw}} \right) \right]^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta = O(\epsilon_s). \tag{L.6}
\end{aligned}$$

Employing that the time derivative of a long-wavelength contribution is small by ϵ_s^2 and that $\nabla_{\mathbf{R}_\perp / \epsilon_\sigma} g^{\text{lw}} \sim \epsilon_\sigma$, we obtain that

$$\begin{aligned}
& \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma} \int B \left[\langle \phi_{\sigma 1}^{\text{sw}} \rangle \left(\partial_t F_{\sigma 1}^{\text{sw}} \right. \right. \\
& \left. \left. + \frac{Z_\sigma \tau_\sigma}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \varphi_0) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} F_{\sigma 1}^{\text{sw}} \right) \right]^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta = O(\epsilon_s). \tag{L.7}
\end{aligned}$$

Now, we use (109) in (L.7), getting

$$\begin{aligned}
& - \sum_\sigma \int B \left[\langle \phi_{\sigma 1}^{\text{sw}} \rangle \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} F_{\sigma 1}^{\text{sw}} \right. \right. \\
& + \frac{u^2}{B} (\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} F_{\sigma 1}^{\text{sw}} \\
& + \frac{\mu}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} F_{\sigma 1}^{\text{sw}} \left. \right)^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta \\
& + \sum_{\sigma, \sigma'} \int B \left[\langle \phi_{\sigma 1}^{\text{sw}} \rangle \left(\mathcal{T}_{NP, \sigma}^* C_{\sigma \sigma'} \left[\mathbb{T}_{\sigma, 0} F_{\sigma 1}^{\text{sw}} \right. \right. \right. \\
& - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma, 0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma, 0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma', 0}^{-1*} F_{\sigma' 0} \left. \right] \\
& + \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{NP, \sigma}^* C_{\sigma \sigma'} \left[\mathcal{T}_{\sigma, 0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma', 0} F_{\sigma' 1}^{\text{sw}} \right. \\
& \left. \left. - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma', 0} \tilde{\phi}_{\sigma' 1}^{\text{sw}} \mathcal{T}_{\sigma', 0}^{-1*} F_{\sigma' 0} \right] \right)^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta = O(\epsilon_s). \tag{L.8}
\end{aligned}$$

Here, we have used the fact that $\langle \phi_{\sigma 1}^{\text{sw}} \rangle$ does not depend on u , and the relations

$$\left[\langle \phi_{\sigma 1}^{\text{sw}} \rangle \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right]^{\text{lw}} = \frac{1}{2} \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \left[\langle \phi_{\sigma 1}^{\text{sw}} \rangle^2 \right]^{\text{lw}} = O(\epsilon_\sigma) \tag{L.9}$$

and

$$\begin{aligned}
& \left[\langle \phi_{\sigma 1}^{\text{sw}} \rangle (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} F_{\sigma 1}^{\text{sw}} \right]^{\text{lw}} = \\
& - \frac{1}{2} \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \times \left[F_{\sigma 1}^{\text{sw}} \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle^2 \right]^{\text{lw}} = O(\epsilon_\sigma). \tag{L.10}
\end{aligned}$$

Finally, to relate (L.8) to (140), we employ that, up to terms of order ϵ_σ ,

$$\int B \left[\langle \phi_{\sigma 1}^{\text{sw}} \rangle \mathcal{T}_{NP, \sigma} C_{\sigma \sigma'}^{\text{sw}} \right]^{\text{lw}} \text{d}u \text{d}\mu \text{d}\theta =$$

$$- \int B \left[\tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{NP,\sigma} C_{\sigma\sigma'}^{\text{sw}} \right]^{\text{lw}} d\mu d\theta, \quad (\text{L.11})$$

where $C_{\sigma\sigma'}^{\text{sw}}(\mathbf{r}, \mathbf{v}, t)$ is the collision operator applied on a function with wavelengths on the order of the sound gyroradius. To prove this, we begin with the particle conservation property of the collision operator, that gives

$$\left[\varphi_1^{\text{sw}}(\mathbf{r}, t) \int C_{\sigma\sigma'}^{\text{sw}}(\mathbf{r}, \mathbf{v}, t) d^3v \right]^{\text{lw}} = 0. \quad (\text{L.12})$$

Using (L.2) this equation becomes

$$\int B \left[\phi_{\sigma 1}^{\text{sw}}(\mathbf{r} - \epsilon_\sigma \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), \mu, \theta, t) \times \right. \\ \left. C_{\sigma\sigma'}^{\text{sw}}(\mathbf{r}, u\hat{\mathbf{b}} + \boldsymbol{\rho}(\mathbf{r}, \mu, \theta) \times \mathbf{B}, t) \right]^{\text{lw}} d\mu d\theta = O(\epsilon_s). \quad (\text{L.13})$$

Since we are only considering the long wavelength component, we can Taylor expand around \mathbf{r} , leaving

$$\int B \left[\phi_{\sigma 1}^{\text{sw}}(\mathbf{r}, \mu, \theta, t) \times \right. \\ \left. C_{\sigma\sigma'}^{\text{sw}}(\mathbf{r} + \epsilon_\sigma \boldsymbol{\rho}(\mathbf{r}, \mu, \theta), u\hat{\mathbf{b}} + \boldsymbol{\rho}(\mathbf{r}, \mu, \theta) \times \mathbf{B}, t) \right]^{\text{lw}} d\mu d\theta \\ = O(\epsilon_s), \quad (\text{L.14})$$

which is equivalent to (L.11).

Substituting (L.11) into (L.8), flux surface averaging and integrating by parts finally yields

$$\sum_\sigma \left\langle \int B \left[F_{\sigma 1}^{\text{sw}} \left(u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right. \right. \right. \\ \left. \left. + \frac{u^2}{B} (\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \hat{\mathbf{b}})) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right. \right. \\ \left. \left. + \frac{\mu}{B} (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} B) \cdot \nabla_{\mathbf{R}_\perp / \epsilon_\sigma} \langle \phi_{\sigma 1}^{\text{sw}} \rangle \right) \right]^{\text{lw}} d\mu d\theta \right\rangle_\psi \\ - \sum_{\sigma, \sigma'} \left\langle \int B \left[\tilde{\phi}_{\sigma 1}^{\text{sw}} \left(\mathcal{T}_{NP,\sigma}^* C_{\sigma\sigma'} \left[\mathbb{T}_{\sigma,0} F_{\sigma 1}^{\text{sw}} \right. \right. \right. \right. \\ \left. \left. - \frac{Z_\sigma \lambda_\sigma}{T_\sigma} \mathbb{T}_{\sigma,0} \tilde{\phi}_{\sigma 1}^{\text{sw}} \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \right. \\ \left. + \frac{\lambda_\sigma}{\lambda_{\sigma'}} \mathcal{T}_{NP,\sigma}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathbb{T}_{\sigma',0} F_{\sigma' 1}^{\text{sw}} \right. \right. \\ \left. \left. - \frac{Z_{\sigma'} \lambda_{\sigma'}}{T_{\sigma'}} \mathbb{T}_{\sigma',0} \tilde{\phi}_{\sigma' 1}^{\text{sw}} \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \right) \right]^{\text{lw}} d\mu d\theta \right\rangle_\psi. \quad (\text{L.15})$$

From this expression and (E.4) we obtain (140) by integrating by parts in μ .

Appendix M. Solvability conditions of the gyrokinetic Fokker-Planck equation of any order

We want to prove that the solvability conditions in subsection 5.2 are the only ones to second order. We do this by showing that, to general order, only the flux-surface averaged zeroth and second moments of the Fokker-Planck equation can give solvability conditions. To order ϵ_s^j the gyrokinetic Fokker-Planck equation for species σ can be written as (we drop the superindex lw in this appendix)

$$\begin{aligned} & \tau_\sigma \lambda_\sigma^{-j} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma j} \\ & - \tau_\sigma \sum_{\sigma'} \left(\mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\lambda_\sigma^{-j} \mathcal{T}_{\sigma,0}^{-1*} G_{\sigma j}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right] \right. \\ & \left. + \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \lambda_{\sigma'}^{-j} \mathcal{T}_{\sigma',0}^{-1*} G_{\sigma' j} \right] \right) = \tau_\sigma \lambda_\sigma^{-j} R_{\sigma j}, \end{aligned} \quad (\text{M.1})$$

where $R_{\sigma j}$ collects terms that do not contain $\langle F_{\sigma j} \rangle$ for any σ . To any order, $\langle G_{\sigma j} \rangle = G_{\sigma j}$ differs from $\langle F_{\sigma j} \rangle$, at most, in terms that have been determined by lowest order equations. We recall that the gyrophase-dependent piece of the distribution function to order $O(\epsilon_s^j)$, $F_{\sigma j} - \langle F_{\sigma j} \rangle$, has been determined by the Fokker-Planck equation of order $O(\epsilon_s^{j-1})$. Also, we point out that we have introduced the factor $\tau_\sigma \lambda_\sigma^{-j}$ in (M.1) because it is convenient for the proof that follows.

We must study the solvability conditions for the set of equations (M.1) when σ runs from 1 to N , with N the number of different species. To this end, it is appropriate to work in the vector space

$$\mathcal{F}^N := \mathcal{F}(\mathcal{P}_1) \times \dots \times \mathcal{F}(\mathcal{P}_\sigma) \times \dots \times \mathcal{F}(\mathcal{P}_N), \quad (\text{M.2})$$

which is the cartesian product of the sets of functions on the phase spaces of the different species. Define $G_j = [G_{1j}, \dots, G_{\sigma j}, \dots, G_{Nj}]$ and $S_j = [\tau_1 \lambda_1^{-j} R_{1j}, \dots, \tau_\sigma \lambda_\sigma^{-j} R_{\sigma j}, \dots, \tau_N \lambda_N^{-j} R_{Nj}] \in \mathcal{F}^N$. On \mathcal{F}^N , the set of equations (M.1) can be rewritten as

$$\mathcal{L}_j G_j = S_j, \quad (\text{M.3})$$

where

$$\begin{aligned} (\mathcal{L}_j G_j)_\sigma &:= \tau_\sigma \lambda_\sigma^{-j} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma j} \\ & - \tau_\sigma \sum_{\sigma'} \left(\mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\lambda_\sigma^{-j} \mathcal{T}_{\sigma,0}^{-1*} G_{\sigma j}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma'0} \right] \right. \\ & \left. + \mathcal{T}_{\sigma,0}^* C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \lambda_{\sigma'}^{-j} \mathcal{T}_{\sigma',0}^{-1*} G_{\sigma' j} \right] \right). \end{aligned} \quad (\text{M.4})$$

The solvability conditions are defined by functions $K \in \mathcal{F}^N$ satisfying

$$\sum_\sigma \int B K_\sigma (\mathcal{L}_j G_j)_\sigma d^3 R du d\mu d\theta = 0 \quad (\text{M.5})$$

for every $G_j \in \mathcal{F}^N$. Then, equation (M.3) implies that

$$\sum_{\sigma} \tau_{\sigma} \lambda_{\sigma}^{-j} \int B K_{\sigma} R_{\sigma j} d^3 R du d\mu d\theta = 0. \quad (\text{M.6})$$

Let us denote by $K = [K_1, \dots, K_{\sigma}, \dots, K_N]$ and $G = [G_1, \dots, G_{\sigma}, \dots, G_N]$ two arbitrary elements of \mathcal{F}^N . Then, a natural scalar product is defined by

$$(K|G) = \sum_{\sigma} \int B(\mathbf{R}) K_{\sigma}(\mathbf{R}, u, \mu, \theta) G_{\sigma}(\mathbf{R}, u, \mu, \theta) d^3 R du d\mu d\theta. \quad (\text{M.7})$$

The question about solvability conditions can be expressed in terms of the scalar product. Our aim is to find those $K \in \mathcal{F}^N$ such that

$$(K|\mathcal{L}_j G_j) = 0 \quad (\text{M.8})$$

for every $G_j \in \mathcal{F}^N$. Since the scalar product is non-degenerate, the condition for K is equivalent to $(\mathcal{L}_j^{\dagger} K|G_j) = 0$, where \mathcal{L}_j^{\dagger} is the adjoint of \mathcal{L}_j . Therefore, the solvability conditions derived to j th order are given by the equations

$$(K|S_j) = 0, \quad K \in \text{Ker}(\mathcal{L}_j^{\dagger}). \quad (\text{M.9})$$

Of course, it might happen that some of these equations be trivial identities that do not add new conditions on lower-order quantities. The important point is that every non-trivial solvability condition is found by calculating all of the equations (M.9).

We turn to compute \mathcal{L}_j^{\dagger} . It is obvious that the piece of \mathcal{L}_j associated to parallel streaming, $\mathcal{L}_{j||}$, defined by

$$(\mathcal{L}_{j||} G_j)_{\sigma} := \tau_{\sigma} \lambda_{\sigma}^{-j} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) G_{\sigma j}, \quad (\text{M.10})$$

is antisymmetric. That is,

$$(K|\mathcal{L}_{j||} G_j) = -(\mathcal{L}_{j||} K|G_j), \quad (\text{M.11})$$

for any $K \in \mathcal{F}^N$. In other words, $\mathcal{L}_{j||}^{\dagger} = -\mathcal{L}_{j||}$. It will also be useful to note that

$$(\mathcal{L}_{j||} G_j)_{\sigma} = F_{\sigma 0} (\mathcal{L}_{j||} \hat{G}_j)_{\sigma}, \quad (\text{M.12})$$

with $\hat{G}_{\sigma j} := G_{\sigma j} / F_{\sigma 0}$, and

$$F_{\sigma 0}(\mathbf{R}, u, \mu) = \frac{n_{\sigma}(\psi)}{(2\pi T_{\sigma}(\psi))^{3/2}} \exp \left(-\frac{\mu B(\psi, \Theta) + u^2/2}{T_{\sigma}(\psi)} \right). \quad (\text{M.13})$$

In order to find the adjoint of the piece of \mathcal{L}_j corresponding to collisions,

$$\mathcal{L}_{j \text{ coll}} = \mathcal{L}_j - \mathcal{L}_{j||}, \quad (\text{M.14})$$

we need to prove a preliminary property. Define

$$\begin{aligned} \hat{C}_{\sigma\sigma'}[\hat{G}_{\sigma j}, \hat{G}_{\sigma' j}] &= C_{\sigma\sigma'} \left[\lambda_{\sigma}^{-j} \mathcal{T}_{\sigma,0}^{-1*} G_{\sigma j}, \mathcal{T}_{\sigma',0}^{-1*} F_{\sigma' 0} \right] \\ &\quad + C_{\sigma\sigma'} \left[\mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \lambda_{\sigma'}^{-j} \mathcal{T}_{\sigma',0}^{-1*} G_{\sigma' j} \right]. \end{aligned} \quad (\text{M.15})$$

From definition (15) one obtains

$$\begin{aligned} \hat{C}_{\sigma\sigma'}[\hat{G}_{\sigma j}, \hat{G}_{\sigma' j}] = \\ \gamma_{\sigma\sigma'} \nabla_{\mathbf{v}} \cdot \int \overset{\leftrightarrow}{\mathbf{W}} (\tau_{\sigma} \mathbf{v} - \tau_{\sigma'} \mathbf{v}') \cdot \\ \left(\frac{\tau_{\sigma}}{\lambda_{\sigma}^j} \nabla_{\mathbf{v}} \hat{g}_{\sigma} - \frac{\tau_{\sigma'}}{\lambda_{\sigma'}^j} \nabla_{\mathbf{v}'} \hat{g}_{\sigma'} \right) f_{\sigma 0}(\mathbf{v}) f_{\sigma' 0}(\mathbf{v}') d^3 v'. \end{aligned} \quad (\text{M.16})$$

Here, to ease the notation, we understand $f_{\sigma 0} \equiv \mathcal{T}_{\sigma,0}^{-1*} F_{\sigma 0}$ and $\hat{g}_{\sigma} \equiv \mathcal{T}_{\sigma,0}^{-1*} \hat{G}_{\sigma j}$. To get (M.16) we have used

$$\nabla_{\mathbf{v}} f_{\sigma 0} = -\frac{1}{T_{\sigma}} \mathbf{v} f_{\sigma 0}, \quad (\text{M.17})$$

$$T_{\sigma} = T_{\sigma'}, \text{ for every pair } \sigma, \sigma', \quad (\text{M.18})$$

and

$$\overset{\leftrightarrow}{\mathbf{W}} (\tau_{\sigma} \mathbf{v} - \tau_{\sigma'} \mathbf{v}') \cdot (\tau_{\sigma} \mathbf{v} - \tau_{\sigma'} \mathbf{v}') \equiv 0, \text{ for every pair } \sigma, \sigma'. \quad (\text{M.19})$$

The operator $\hat{C}_{\sigma\sigma'}[\hat{G}_{\sigma j}, \hat{G}_{\sigma' j}]$ does not have nice symmetry properties with respect to the scalar product, but its symmetrization in σ and σ' does. A simple integration by parts yields the following symmetric expression for any pair of functions $k_{\sigma}(\mathbf{v})$ and $k_{\sigma'}(\mathbf{v})$:

$$\begin{aligned} \tau_{\sigma} \int k_{\sigma}(\mathbf{v}) \mathcal{T}_{\sigma,0}^* \hat{C}_{\sigma\sigma'}[\hat{G}_{\sigma j}, \hat{G}_{\sigma' j}] d^3 v \\ + \tau_{\sigma'} \int k_{\sigma'}(\mathbf{v}) \mathcal{T}_{\sigma',0}^* \hat{C}_{\sigma'\sigma}[\hat{G}_{\sigma' j}, \hat{G}_{\sigma j}] d^3 v = \\ -\gamma_{\sigma\sigma'} \int (\tau_{\sigma} \nabla_{\mathbf{v}} k_{\sigma} - \tau_{\sigma'} \nabla_{\mathbf{v}'} k_{\sigma'}) \cdot \overset{\leftrightarrow}{\mathbf{W}} (\tau_{\sigma} \mathbf{v} - \tau_{\sigma'} \mathbf{v}') \cdot \\ (\tau_{\sigma} \nabla_{\mathbf{v}} (\lambda_{\sigma}^{-j} \hat{g}_{\sigma}) - \tau_{\sigma'} \nabla_{\mathbf{v}'} (\lambda_{\sigma'}^{-j} \hat{g}_{\sigma'})) f_{\sigma 0}(\mathbf{v}) f_{\sigma' 0}(\mathbf{v}') d^3 v d^3 v'. \end{aligned} \quad (\text{M.20})$$

Hence, denoting $K_{\sigma} \equiv \mathcal{T}_{\sigma,0}^* k_{\sigma}$, we can write

$$\begin{aligned} \tau_{\sigma} \int k_{\sigma}(\mathbf{v}) \mathcal{T}_{\sigma,0}^* \hat{C}_{\sigma\sigma'}[\hat{G}_{\sigma j}, \hat{G}_{\sigma' j}] d^3 v \\ + \tau_{\sigma'} \int k_{\sigma'}(\mathbf{v}) \mathcal{T}_{\sigma',0}^* \hat{C}_{\sigma'\sigma}[\hat{G}_{\sigma' j}, \hat{G}_{\sigma j}] d^3 v = \\ \frac{\tau_{\sigma}}{\lambda_{\sigma}^j} \int \mathcal{T}_{\sigma,0}^* \hat{C}_{\sigma\sigma'} [\lambda_{\sigma}^j K_{\sigma}, \lambda_{\sigma'}^j K_{\sigma'}] \hat{g}_{\sigma}(\mathbf{v}) d^3 v \\ + \frac{\tau_{\sigma'}}{\lambda_{\sigma'}^j} \int \mathcal{T}_{\sigma',0}^* \hat{C}_{\sigma'\sigma} [\lambda_{\sigma'}^j K_{\sigma'}, \lambda_{\sigma}^j K_{\sigma}] \hat{g}_{\sigma'}(\mathbf{v}) d^3 v. \end{aligned} \quad (\text{M.21})$$

Thus, for every $K \in \mathcal{F}^N$,

$$\begin{aligned} (K | \mathcal{L}_j G) = \\ - \sum_{\sigma} \int \left[\frac{\tau_{\sigma} B}{\lambda_{\sigma}^j F_{\sigma 0}} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) F_{\sigma 0} K_{\sigma} \right] G_{\sigma j} d^3 R du d\mu d\theta \end{aligned}$$

$$-\sum_{\sigma,\sigma'} \int \frac{\tau_\sigma B}{\lambda_\sigma^j F_{\sigma 0}} \mathcal{T}_{\sigma,0}^* \hat{C}_{\sigma\sigma'} [\lambda_\sigma^j K_\sigma, \lambda_{\sigma'}^j K_{\sigma'}] G_{\sigma j} d^3 R du d\mu d\theta. \quad (\text{M.22})$$

This means that for any $K \in \mathcal{F}^N$, the action of the adjoint of \mathcal{L}_j is given by

$$\begin{aligned} (\mathcal{L}_j^\dagger K)_\sigma &= - \left[\frac{\tau_\sigma}{\lambda_\sigma^j F_{\sigma 0}} \left(u \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} - \mu \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} B \partial_u \right) F_{\sigma 0} K_\sigma \right] \\ &\quad - \sum_{\sigma'} \frac{\tau_\sigma}{\lambda_\sigma^j F_{\sigma 0}} \mathcal{T}_{\sigma,0}^* \hat{C}_{\sigma\sigma'} [\lambda_\sigma^j K_\sigma, \lambda_{\sigma'}^j K_{\sigma'}]. \end{aligned} \quad (\text{M.23})$$

An entropy argument similar to the one employed in subsection 3.2 can be used here to obtain the solutions of $\mathcal{L}_j^\dagger K = 0$. Multiplying the equation $\mathcal{L}_j^\dagger K = 0$ by $\lambda_\sigma^j B K_\sigma F_{\sigma 0}$, integrating over u, μ , and θ , flux-surface averaging, and summing over all of the species gives

$$- \left\langle \sum_{\sigma,\sigma'} \tau_\sigma \int B K_\sigma \mathcal{T}_{\sigma,0}^* \hat{C}_{\sigma\sigma'} [\lambda_\sigma^j K_\sigma, \lambda_{\sigma'}^j K_{\sigma'}] du d\mu d\theta \right\rangle_\psi = 0, \quad (\text{M.24})$$

that can be recasted into

$$\begin{aligned} &\left\langle \sum_{\sigma,\sigma'} \frac{\gamma_{\sigma\sigma'}}{2} \int (\tau_\sigma \nabla_{\mathbf{v}} k_\sigma - \tau_{\sigma'} \nabla_{\mathbf{v}'} k_{\sigma'}) \cdot \overleftrightarrow{\mathbf{W}} (\tau_\sigma \mathbf{v} - \tau_{\sigma'} \mathbf{v}') \right. \\ &\quad \left. (\tau_\sigma \nabla_{\mathbf{v}} k_\sigma - \tau_{\sigma'} \nabla_{\mathbf{v}'} k_{\sigma'}) f_{\sigma 0} f_{\sigma' 0} d^3 v d^3 v' \right\rangle_\psi = 0. \end{aligned} \quad (\text{M.25})$$

This equation has the following types of solutions: $k_\sigma = q_\sigma(\mathbf{r})$, $k_\sigma = \tau_\sigma^{-1} \mathbf{V}(\mathbf{r}) \cdot \mathbf{v}$, and $k_\sigma = Q(\mathbf{r}) \mathbf{v}^2/2$. Again, in analogy with the calculation of subsection 3.2, it is easy to show that $\mathcal{L}_j^\dagger K = 0$ implies that $\mathbf{V}(\mathbf{R}) \equiv 0$ and that $\{q_\sigma, \sigma = 1, \dots, N\}$ and Q are flux functions, but otherwise arbitrary. In other words, every $K \in \text{Ker}(\mathcal{L}_j^\dagger)$ can be written as a linear combination of elements of the form

$$\begin{aligned} &[q_1(\psi), 0, \dots, 0], \dots, [0, \dots, q_\sigma(\psi), \dots, 0], \dots, [0, \dots, q_N(\psi)], \\ &Q(\psi) (u^2/2 + \mu B) [1, \dots, 1, \dots, 1], \end{aligned} \quad (\text{M.26})$$

where the functions $\{q_\sigma, \sigma = 1, \dots, N\}$ and Q are arbitrary. Then, the solvability conditions are given by (M.9). Equivalently, due to the arbitrariness of the functions $\{q_\sigma, \sigma = 1, \dots, N\}$ and Q , the solvability conditions can be expressed as

$$\sum_\sigma \left\langle \tau_\sigma \lambda_\sigma^{-j} \int B K_\sigma R_{\sigma j} du d\mu d\theta \right\rangle_\psi = 0, \quad K \in \text{Ker}(\mathcal{L}_j^\dagger). \quad (\text{M.27})$$

More concretely, all the solvability conditions of the Fokker-Planck equation to order $O(\epsilon_s^j)$ are obtained by working out

$$\begin{aligned} &\left\langle \tau_\sigma \lambda_\sigma^{-j} \int B R_{\sigma j} du d\mu d\theta \right\rangle_\psi = 0, \text{ for each } \sigma, \text{ and} \\ &\left\langle \sum_\sigma \tau_\sigma \lambda_\sigma^{-j} \int B (u^2/2 + \mu B) R_{\sigma j} du d\mu d\theta \right\rangle_\psi = 0. \end{aligned} \quad (\text{M.28})$$

The proof in this appendix guarantees that transport equations for particle and total energy density are the only solvability conditions for the long-wavelength second-order Fokker-Planck equation, (119). Finally, the reader can immediately check that when (M.28) is applied to the first-order equations, (105), no condition is obtained.

References

- [1] Catto P J 1978 *Plasma Phys.* **20** 719
- [2] Dimits A M, Williams T J, Byers J A and Cohen B I 1996 *Phys. Rev. Lett.* **77** 71
- [3] Dorland W, Jenko F, Kotschenreuther M and Rogers B N 2000 *Phys. Rev. Lett.* **85** 5579
- [4] Dannert T and Jenko F 2005 *Phys. Plasmas* **12** 072309
- [5] Candy J and Waltz R E 2003 *J. Comput. Phys.* **186** 545
- [6] Chen Y and Parker S E 2003 *J. Comput. Phys.* **189** 463
- [7] Peeters A G et al 2009 *Comput. Phys. Commun.* **180** 2650
- [8] Frieman E A and Chen L 1983 *Phys. Fluids* **25** 502
- [9] Lee X S, Myra J R and Catto P J 1983 *Phys. Fluids* **26** 223
- [10] Lee W W 1983 *Phys. Fluids* **26** 556
- [11] Bernstein I B and Catto P J 1985 *Phys. Fluids* **28** 1342
- [12] Parra F I and Catto P J 2008 *Plasma Phys. Control. Fusion* **50** 065014
- [13] Dubin D H E, Krommes J A, Oberman C and Lee W W 1983 *Phys. Fluids* **26** 3524
- [14] Hahm T S 1988 *Phys. Fluids* **31** 2670
- [15] Brizard A J and Hahm T S 2007 *Rev. Mod. Phys.* **79** 421
- [16] Parra F I and Calvo I 2011 *Plasma Phys. Control. Fusion* **53** 045001
- [17] Krommes J A 2012 *Annu. Rev. Fluid Mech.* **44** 175
- [18] Parra F I and Catto P J 2009 *Plasma Phys. Control. Fusion* **51** 095008
- [19] Parra F I and Catto P J 2009 *Phys. Plasmas* **16** 124701
- [20] Parra F I and Catto P J 2010 *Phys. Plasmas* **17** 056106
- [21] Parra F I and Catto P J 2010 *Plasma Phys. Control. Fusion* **52** 085011
- [22] Parra F I and Catto P J 2010 *Plasma Phys. Control. Fusion* **52** 045004
- Parra F I and Catto P J 2010 *Plasma Phys. Control. Fusion* **52** 059801
- [23] Parra F I, Barnes M and Peeters A G 2011 *Phys. Plasmas* **18** 062501
- [24] Parra F I, Barnes M and Catto P J 2011 *Nucl. Fusion* **51** 113001
- [25] Parra F I, Barnes M, Calvo I and Catto P J 2012 *Phys. Plasmas* **19** 056116
- [26] Hinton F L and Hazeltine R D 1976 *Rev. Mod. Phys.* **48** 239
- [27] Helander P and Sigmar D J 2002 *Collisional Transport in Magnetized Plasmas* (*Cambridge Monographs on Plasma Physics*) ed Haines M G et al (Cambridge, UK: Cambridge University Press)
- [28] Catto P J and Simakov A N 2005 *Phys. Plasmas* **12** 012501
- Catto P J and Simakov A N 2005 *Phys. Plasmas* **12** 114503
- [29] Wong S K and Chan V S 2009 *Phys. Plasmas* **16** 122507
- [30] Kovrizhnykh L M 1969 *Soviet Physics JETP* **29** 475
- [31] Rutherford P H 1970 *Phys. Fluids* **13** 482
- [32] Sugama H, Okamoto M, Horton W and Wakatani M 1996 *Phys. Plasmas* **3** 2379
- [33] Chapman S and Cowling T G 1970 *The Mathematical Theory of Non-uniform Gases*, 3rd edition (*Cambridge Mathematical Library*) (Cambridge, UK: Cambridge University Press)
- [34] Braginskii S I 1965 in *Reviews of Plasma Physics* ed Leontovich M A (Consultants Bureau, New York, 1965)
- [35] McKee G R et al 2001 *Nucl. Fusion* **41** 1235
- [36] Barnes M, Parra F I and Schekochihin A A 2011 *Phys. Rev. Lett.* **107** 115003
- [37] Littlejohn R G 1983 *J. Plasma Physics* **29** 111

- [38] Dimits A M, LoDestro L L and Dubin D H E 1992 *Phys. Fluids B* **4** 274
- [39] Dimits A M 2012 *Phys. Plasmas* **19** 022504
- [40] D'haeseleer W D, Hitchon W N G, Callen J D and Shohet J L 1991 *Flux Coordinates and Magnetic Field Structure* ed Glowinski R *et al* (Berlin, Heidelberg, Germany: Springer-Verlag)
- [41] Lenard A 1960 *Ann. Phys.* **10** 390
- [42] Littlejohn R G 1981 *Phys. Fluids* **24** 1730